2-4 DIGITAL COMPUTER TECHNIQUES FOR GLOW PROBLEMS

There are four broad classes of groundwater models which are (1) physical, (2) analog, (3) analytycal formula, (4) numerical, which include the finite difference and the finite element. The first two models are not presently favored . Analytical formulas are widely used for flow problems. They require only paper, pencil, and tables of well function, due to idealized aquifer and boundary conditions. With the aid of a pocket-computer, they can be applied to wide areas.

In more compex hydrogeological conditions, numerical models must be used to avoid inaccuracy. There are many numerical techniques which include the finite difference (FD) and the finite element (FE). They only differ from one another in the way the differential equations are approximated and solved with a digital computer. Comparision between the FD and the FE is shown in Table 2-1. Each technique has its advantages and disadvantages.

2-4-1 FINITE DIFFERENCE METHOD

(1) Governing Equation

The partial differential equation which governs the non-steady state two-dimensional flow of groundwater in an artesian nonhomogeneous and isotropic aquifer, can be stated as follows :

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right)=S\frac{\partial h}{\partial t}+W \qquad \cdots\cdots\cdots\cdots \quad (21)$$

where,

T : transmissivity (L^2T^{-1})

S : storativity (dimensionless)

h : artesian head (L)

x,y: cartesian coordinates (L)

w : source or sink (L T^{-1})

t : time (T)

Table 2-1 Comparsion between FD and FE

Attribute	FD	FE
1 Code Availability	Excellent	> Good
2 Convenience	Mesh generation relatively automatic data handling can be easy	> by hand or automatic
3 Accuracy	Mesh design roughly approximates boundary and location of well	< Mesh can follow boundary condition, observation well, and pumping well
4 Flexibility	Easier to modify	> not easy
5 Code Efficency	Usually in-core	$\gtrless$ in-core or out-core

(2) Finite Difference Approximation

In the finite difference technique, space and time variables are treated as discrete parameters. Firstly, the aquifer investigated is subdivided into rectangular blocks by the grid system. These blocks have volume $\Delta x \ \Delta ym$ where "m" is the thickness of the aquifer. (Figure 2-8)

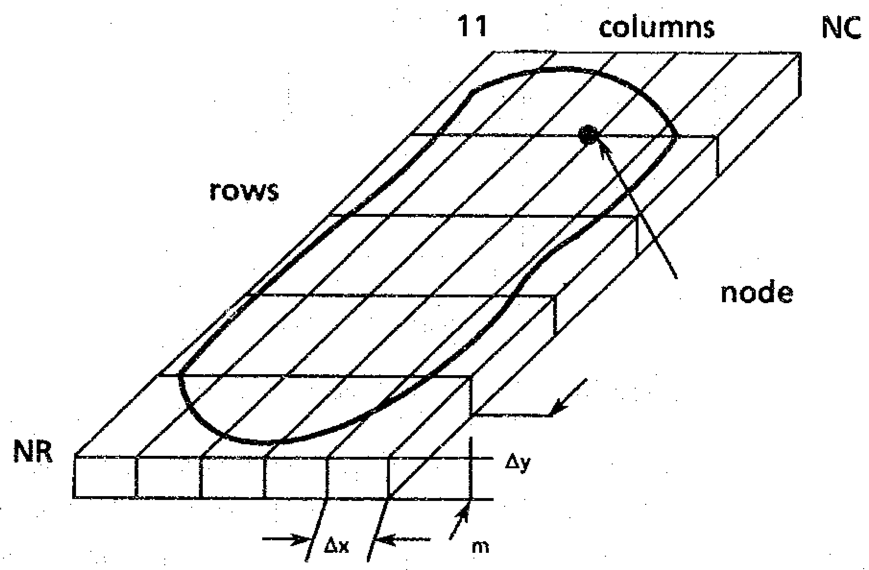


Figure 2-8 Grid System

The differentials ∂x and ∂y are approximated by the finite lengths Δx and Δy, respectively. The area Δx and Δy should be small compared with the total area of the aquifer, so that the discrete model reasonable represents the continuous aquifer.

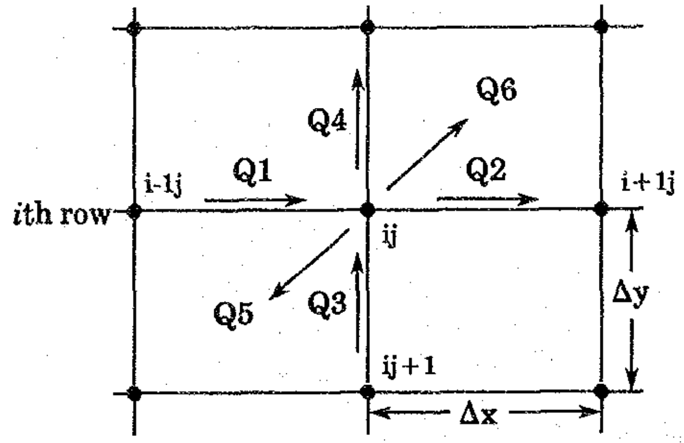


Figure 2-9 Finite Difference Grid

The groundwater flow in the aquifer is approximated by the flow between nodes. Flow rate terms Q1, Q2, Q3, . . . Q6 are arbitrarily assigned flow directions as illustrated in Figure 2-9. Q1, Q2, Q3 and Q4 represent node-to-node water transfer rates. Q5 is the flow rate associated with the amount of water taken into or released from storage per unit time increment Δt. Q6 is defined as a net withdrawal rate and represents source or sink term W of Eq. (11). The conservation of mass requires that the flow rates entering and leaving the node ij are equal as follows :

$$Q1 + Q3 = Q2 + Q4 + Q5 + Q6 \qquad (22)$$

Determining the values of the flow rate terms of Eq. (22) involves three considerations. First, it is necessary to define what portion of the aquifer is represented by each individual term. Secondly, it must be kept in mind that, although the flow rates may take place in any direction in the aquifer system, they are restricted to the x and y directions in the finite difference approach. The portions of the aquifer included in the flow rate terms then may be referred to as 'vector volumes' to emphasize that not only a volume but also the direction of flow is being considered. Finally, since time is discretized, Eq. (22) represents an instantaneous balance at the end of a time increment.

(3) Derivation of Approximate Equation

Horizontal projections of the vector volume of the node-to-node flow rate terms, Q1, Q2, Q3 and Q4, are defined as illustrated in Figure-11. All Vector volumes of Figure-12 have a vertical dimension extending the full depth of the aquifer, m. Furthermore, the portion of aquifer involved with each of these flow rate terms extends in width one-half of the grid interval of either side of the line between node points, and is equal in length to the grid interval. Darcy's Law is then applied to the flow rate terms, Q1 through Q4, to give

$$Q1 = T_{i-1,j,2}\,(h_{i-1,j} - h_{i,j})\,\Delta y / \Delta x \quad (23a)$$
$$Q2 = T_{i,j,2}(h_{i,j} - h_{i+1,j})\,\Delta y / \Delta x \quad (23b)$$
$$Q3 = T_{i,j,1}\,(h_{i,j+1} - h_{i,j})\,\Delta x / \Delta y \quad (23c)$$
$$Q4 = T_{i,j-1,1}\,(h_{i,j} - h_{i,j-1})\,\Delta x / \Delta y \quad (23d)$$

where, $T_{i,j,1}$: aquifer transmissivity within the vector volume between nodes i, j, and i, j + 1 (see Figures 2-10c and d)

$T_{i,j,2}$: aquifer transmissivity within the vector volume between nodes i, j, and i + 1, j (see Figure 2-10a and b)

$h_{i,j}$: calculated heads at the end of a time increment measured from an arbitrary reference level at node i, j

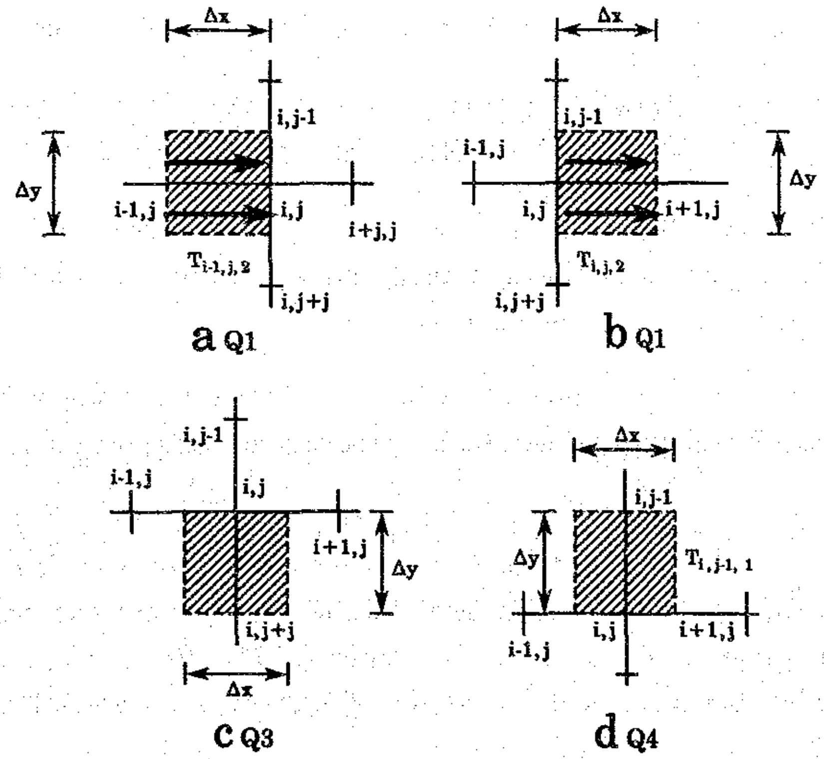


Figure 2-10 Vector Volumes for Node-to-Node Flow Rate Terms

Horizontal projections of the vector volumes of the flow terms Q5 and Q6, extend the full depth of the aquifer and have horizontal dimensions of Δx and Δy, the volumes being centered around the node point i, j.

The flow rate terms Q5, representing the rate at which water is taken into storage, is given by

$$Q5 = S\,\Delta x\,\Delta y\,(h_{i,j}\ h0_{i,j}) / \Delta t \tag{24}$$

where, $h0_{i,j}$: calculated head at node i, j at the end of the previous time increment Δt

Δt : time increment elapsed since last calculation of heads

The flow rate term Q6 is made equal to a net withdrawal rate from the vector volume of node i, j of Figure-11 as follows,

$$Q6 = Q1, j \tag{25}$$

Substitution of Equations (23), (24) and (25) into Equation (22) results in

$$T_{i-1,j,2}(h_{i-1,j} - h_{i,j})\,\Delta y / \Delta x + T_{i,j,1}(h_{i,j+1} - h_{i,j})\,\Delta x / \Delta y = T_{i,j,2}\,(h_{i,j} - h_{i+1,j})\,\Delta y / \Delta x + T_{i,j-1,1}\,(h_{i,j} - h_{i,j-1})\,\Delta x / \Delta y + S\,\Delta x \Delta y\,(h_{i,j} - h0_{i,j}) / \Delta t + Q_{i,j}$$

Dividing both sides of Equation (26) by the product of $\Delta x\ \Delta y$, yields

$$T_{i-1,j,2}\,(h_{i-1,j} - h_{i,j}) / \Delta x^2 + T_{i,j,2}\,(h_{i+1,j} - h_{i,j}) / \Delta x^2 + T_{i,j,1}(h_{i,j+1} - h_{i,j}) / \Delta y^2 + T_{i,j-1,1}\,(h_{i,j-1} - h_{i,j}) / \Delta y^2 = S\,(h_{i,j} - h0_{i,j}) / \Delta t + Q_{i,j} / \Delta x \Delta y \tag{27}$$

Equation (27) is the finite difference form of the partial differential equation (see Equation (21)) governing the nonsteady state, two-dimensional flow of groundwater in an artesian, nonhomogeneous aquifer.

Since an equation of the same form as Equation (27), is constructed for every node, a set of simultaneous equations should be solved for the principle unknown $h_{i,j}$.

The way of deriving the finite difference equation shown here, is based on physical standpoint involving Darcy's Law and the Principle of Conservation of Mass, which was given by Prickett and Lonnquist (1971). Remson et al. (1971) give rather complete mathematical derivations of finite difference equations.

2-4-2 FINITE ELEMENT METHOD

(1) Governing Equation

The governing differential equation for two-dimensional, essentially horizontal groundwater flow in a non-homogeneous, isotropic, aquifer with leakage is (see Figure 2-11)

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right)+Q+\frac{k'}{b'}(H-h)=S\frac{\partial h}{\partial t} \qquad (28\text{-a})$$

where
- T : transmissivity (L/T)
- S : strativity (1)
- h : artesian heads : solution variables (L)
- x,y : Cartesian coordinates (L)
- Q : net flux into the aquifer from point or distributed sources (and skins) (L/T)
- k' : vertical permeability of the aquitard above the aquifer (L/T)
- b' : thickness of the aquitard
- H : piezometric head in a vertically ajacent aquifer separated by the aquitard (L)
- t : time (T)

and boundary conditions are (see Figure 2-12)

$$\frac{\partial h}{\partial x}=V \qquad \text{on} \quad C1 \qquad (28\text{-b})$$

$$h=HC \qquad \text{on} \quad C2 \qquad (28\text{-c})$$

in which
- V : flux into aquifer from uncomputed area
- Hc : constant head
- n : outward normal vector to the boundary

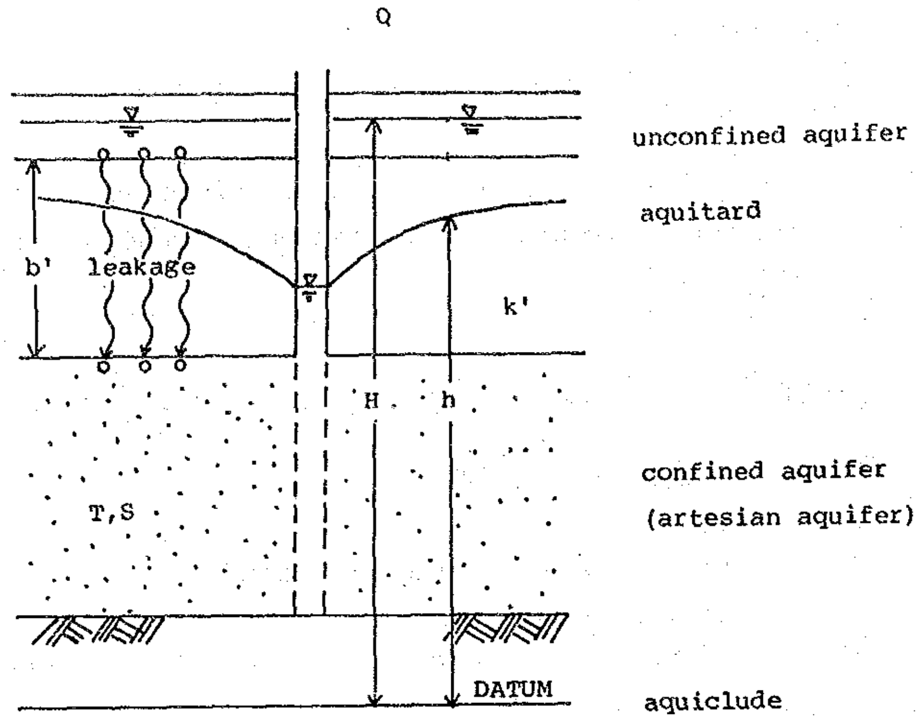


Fig. 2-11 Definition Sketch

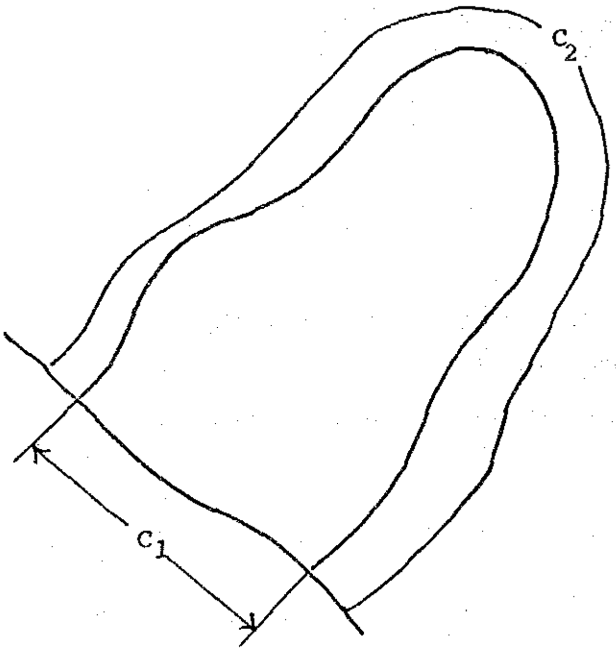


Fig. 2-12 Boundary Definitions

(2) Finite Element Formulation

Equation (28-a) is solved by dividing the aquifer domain into elements and points as illustrated in Figure 2-12.
In the divided triangular finite element the unknown artesian head can be approximated by the combination of linear shape function Ni (x,y) as

$$h = N_1 h_1 + N_2 H_2 + N_3 h_3 \qquad (29)$$

$$\left.\begin{aligned} N_1 &= \frac{a_1 + b_1 x + c_1 y}{2} \\ N_2 &= \frac{a_2 + b_2 x + c_2 y}{2} \\ N_3 &= \frac{a_3 + b_3 x + c_3 y}{2} \end{aligned}\right\} \qquad (30)$$

$a_1 = x_2 y_3 - x_3 y_2$

$a_2 = x_3 y_1 - x_1 y_3$

$a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_1 - y_3$

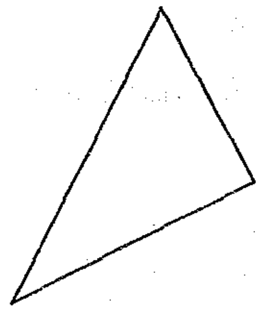
$b_2 = y_3 - y_1$

$b_3 = y_1 - y_2$

$c_1 = X_3 - X_2$

$c_2 = X_1 - X_3$

$c_3 = X_2 - X_1$



where h : unknown artesian heads in a triangular finite element

h_1, h_2, h_3 : artesian heads at each corner of a triangular finite element

Δ : area of a triangular finite element

xi, yi : cartesian coordinates at each corner of a triangular finite element (i = 1, 2, 3)

According to the Galerkin Method, the weighted and integrated equation residual is equal to zero. Therefore, Equation (28-a) is rearranged as follows:

$$\iint [N]^T [Te \frac{\partial [N]}{\partial x} \{he\} + Te \frac{\partial [N]}{\partial y} \{he\} + Qe + \frac{k'}{b'} [N] \{He\} - \frac{k'}{b'} [N] \{he\} - Se \frac{\partial [N]}{\partial t} \{he\}] d\Omega e = 0 \quad (31)$$

In which superscript(e) denotes the particular element under consideration.

$$[N] = [N_1 \; N_2 \; N_3]$$

$$[N]^T = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

$$\{h\} = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix}$$

Using Green's First Theorem and applying the boundary conditions (28-b) and (28-c), we have

$$\iint [Te \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + Te \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} + \frac{ke}{be} [N]^T [N] + \frac{ke'}{be'} [N]^T [N] + \frac{Se}{\Delta t} [N]^T [N] \{he\}] d\Omega e = \iint [N]^T Qe \, d\Omega e$$

$$+ \iint \frac{ke'}{be'} [N]^T [N] \{He\} d\Omega e + \iint \frac{Se}{\Delta t} [h]^T [N] d\Omega e \quad (32)$$

Therefore, Equation (32) is rearranged as follows:

$$[Ke] \{he\} = \{fe\} \quad (33)$$

in which

$$[Ke] = Te \iint \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} d\Omega e + Te \iint \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} d\Omega e + \frac{ke'}{be'} + \frac{ke'}{be'} \iint [N]^T [N] d\Omega e \quad (34)$$

$$\{fe\} = -\iint [N]^T Qe \, d\Omega e + \iint \frac{ke'}{be'} [N]^T [N] d\Omega e \iint \frac{Se}{\Delta t} [N]^T [N] \{hoe\} d\Omega e \quad (35)$$

where [ke] : element "stiffness" matrix

{fe} : element "force" vector

We can calculate according to the equation derived so far.

$$\iint Te \frac{\partial [N]^T}{\partial x} \frac{\partial [h]}{\partial x} \, d\Omega e$$

$$= Te \cdot \begin{pmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{pmatrix} \cdot \Delta$$

$$= Te \cdot \begin{pmatrix} \frac{b_1}{2\Delta} \\ \frac{b_2}{2\Delta} \\ \frac{b_3}{2\Delta} \end{pmatrix} \begin{pmatrix} \frac{b_1}{2\Delta} & \frac{b_2}{2\Delta} & \frac{b_3}{2\Delta} \end{pmatrix} \cdot \Delta$$

$$= \frac{Te}{4\Delta} \begin{pmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{pmatrix}$$

In the same way, we have

$$\iint Te \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \, d\Omega e$$

$$= \frac{Te}{4\Delta} \begin{pmatrix} c_1 c_1 & c_1 c_2 & c_1 c_3 \\ c_2 c_1 & c_2 c_2 & c_2 c_3 \\ c_3 c_1 & c_3 c_2 & c_3 c_3 \end{pmatrix}$$

In order to evaluate the integral

$$\iint [N]^T [N] \, d\Omega e \qquad (37)$$

We can use the formula

$$\iint N_1^i N_2^j N_3^k \, d\Omega e = \frac{i!j!k!}{(i!+j!+k!+2)} 2\Delta \qquad (38)$$

Integration Formula (38) is valid for two-dimensional elements.
Using Formula (38), the integral

$$\iint [N]^T [N] \, d\Omega e \qquad (39)$$

is as follows :

$$\iint [N]^T [N] \, d\Omega e = \begin{pmatrix} \iint N_1^2 d\Omega e & \iint N_1 N_2 \, d\Omega e & \iint N_1 N_3 \, d\Omega e \\ \iint N_2 N_1 d\Omega e & \iint N_2^2 d\Omega e & \iint N_2 N_3 \, d\Omega e \\ \iint N_3 N_1 d\Omega e & \iint N_3 N_2 d\Omega e & \iint N_3^2 d\Omega e \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} \Delta \qquad (40)$$

Assembling the element "stiffness" matrix and "force" vector, we obtain

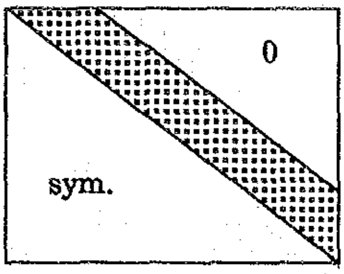
$$[K] \{h\} = \{f\} \qquad (41)$$

Where $[K] = \sum_e [Ke]$: the global "stiffness" matrix
$\{f\} = \sum_e \{fe\}$: the global "force" vector
$\{h\} = \sum_e \{he\}$

Solving above the matrix equation, we obtain unknown heads.

(3) Soltion of the Matrix Equation

The systems of equations obtained in the most practical problems are not only symmetric but also banded matrix which may be written as

A= 

In this case, we can solve the matrix equations quickly and reduce the capacity of the computer memory.

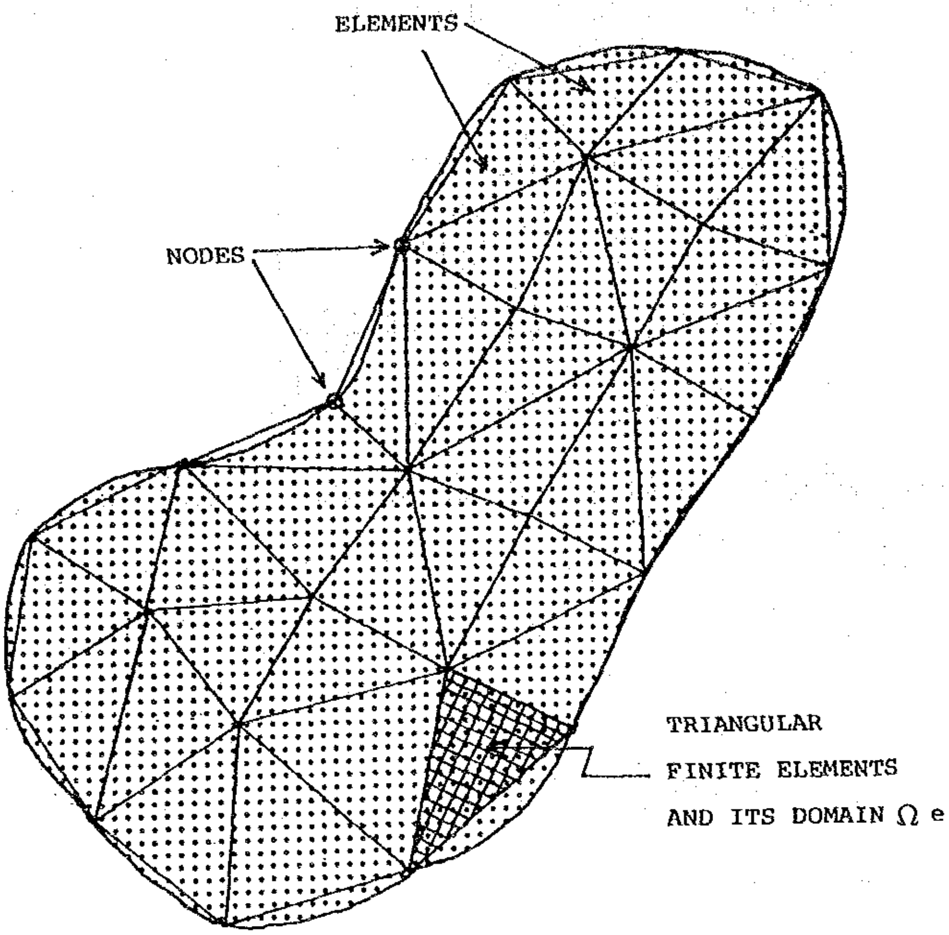
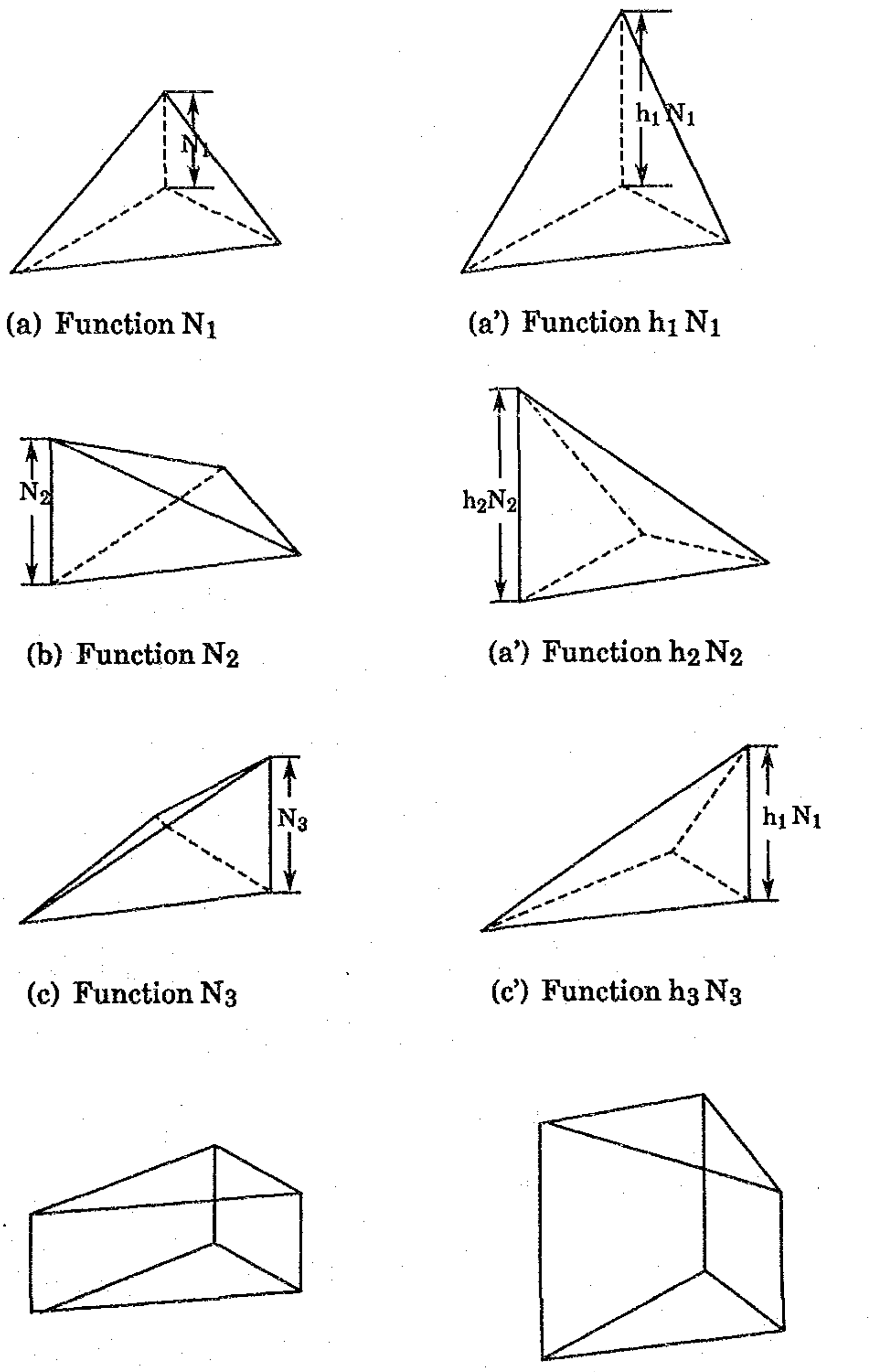


Fig. 2-13 Example of Aquifer Domain



(a) Function N_1

(a') Function $h_1 N_1$

(b) Function N_2

(a') Function $h_2 N_2$

(c) Function N_3

(c') Function $h_3 N_3$

(d) Function $N_1+N_2+N_3=1$

(d') Function $h=h_1 N_1+h_2 N_2+h_3 N_3$

Figure 2-14 Linear Shape Function for Triangular Elements