FUNDAMENTALS OF GRAVITATIONAL METHOD OF APPLIED GEOPHYSICS

JAPAN INTERNATIONAL COOPERATION AGENCY

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BY

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JAPAN INTERNATIONAL COOPERATION AGENCY

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Preface

In December 1975 I was appointed visiting Professor of Applied Geophysics in the Department of Geology, University of Ibadan, with the aid of Technical Assistance of the Government of Japan. Since then I have given a course of lectures on the gravitational and magnetic method of applied geophysics to the final year students, in addition to the supervision of postgraduate research.

The lectures are intended to present the fundamentals of the geophysical methods in easily understandable form. This is the book printed from the cyclostyled volume for the gravitational method, the copies of which have been handed to the students as the 'handout'. The volume was written with reference to the books listed at the end of this volume, most of which were available in the Department. I owe a great deal to these books.

At this opportunity, I wish to express my sincere thanks to the staff members of the Department, who have been so friendly that I have worked here pleasantly without any difficulty, especially to Professor M.O. Oyawoye, former Head of Geology Department, who has not only helped me in doing my official work, but also privately considered my family so that we spend good time in Ibadan, Prof. E.A. Fayose, present Head of Geology. Department, who has always been well disposed to me and to Dr. O. Ofrey, and Dr. C.I. Adighije, both lecturers of Geophysics of the Department, who have assisted me in every respect of my duty, particularly, in the practical training of the students, and in guiding them for their B.Sc. theses.

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CONTENT

Gravitational Method

by

H. Higasinaka

I. Gravity measurement

1. Gravity

If the earth consists of concentric spherical shells each of which has a constant density, and is not rotating, its attraction is directed toward its centre and must be the same at everywhere on its surface.

However, since the earth in actually rotating and has a property of liquid more or less, it has, owing to the centrifugal force, a form of an oblate ellipsoid compressed to the direction of its rotation axis. The form is an equilibrium one resulted from the resultant of the attraction and the centrifugal force.

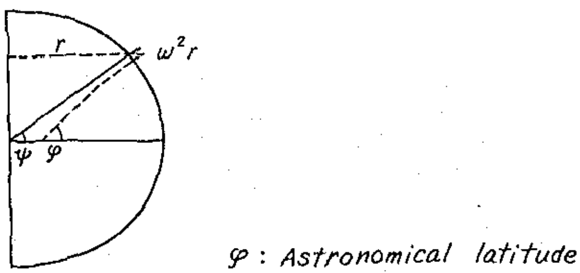


Fig. 1

The resultant force acting on a unit mass at a point is called the gravity or the force of gravity at that point. When we consider the value without regard to the mass of that point, it becomes the acceleration of gravity, which is indicated by g. The word gravity is often used for the acceleration as well as the force.

If the earth is assumed to be a uniform sphere or a sphere consisting of uniform shells, its attraction F on a mass of m gr. at its surface is given by

$$F = G \frac{mE}{R^2} \quad \frac{cm \ gr}{sec^2} ,$$

where G is the gravitational constant, E the total mass of the earth, and R its radius. According to the determination by P.R. Heyl,

$$G = 6.670 \times 10^{-8} \quad \frac{cm^3}{gr. \ sec^2} \quad \text{in 1930}$$

$$G = 6.673 \times 10^{-8} \quad \text{"} \quad \text{in 1942.}$$

The centrifugal force acting on the mass m is represented by mw^2r, where w is the angular velocity of the rotation of the earth, and

$r = R \cos\psi$,

ψ = the latitude of that point. Here, can be used instead of Ψ.

$w = 2\pi / 86164.09 = 7.29212 \times 10^{-5} \ sec^{-1}$

R = 6371 km.

The force vanishes at the poles and attains a maximum value 3.4 dynes for a unit mass at the equator. Thus the gravity is maximum at the poles and minimum at the equator. The difference of gravity at the poles and the equator increases by the effect due to the oblate form of the earth.

As the effect of the centrifugal force is small, compared with the earth's attraction, i.e. 0.35% of the latter as the maximum, only the attraction is often called the gravity. The attraction F upon a mass of 1 gr. at the earth's surface is approximately given:

$$F = G \frac{1}{R^2} \quad 4/3\,\pi R^3 \rho$$

$$= 6.670 \times 10^{-8} \times 4/3. \ \pi \ .6370 \times 10^5. \ 5.5$$

$$= 980 \text{ dyne},$$

where ρ is the mean density of the earth and is taken 5.5.

For the unit of force, dyne ($gr.cm.sec^{-2}$) is used, while for the acceleration gal ($cm.sec^{-2}$) is used as the unit after the name of Galileo. 0.001 gal is called milligal (abbr. mgal).

Gravity changes with time. Its periodic variation may be caused by the following factors: the apparent change of the positions of the moon and the sun, their tidal effect, change of the earth's attraction due to the variation of latitude, migration of atomosphere, etc. Its secular variation may be occurred by geological changes such as upheaval and subsidence of the crust or its part, erosion and deposition, earthquakes and volcanic eruptions, etc.

These effects are so small that even the largest one, caused by the motion of the moon and the sun, attains to only 10^{-7} of the normal field, if a relatively short interval is concerned.

However, in the case of geophysical prospecting for detecting detailed geological structures, the effect of variation which can be measured, should be taken into account as the correction to the observed gravity value. In addition, for the prediction of earthquakes or volcanic eruptions, the detection of small change of gravity may be considered useful.

2. Relative measurement

The period of free oscillation T of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{mgh}} \qquad \text{or} \qquad g = \frac{4\pi^2 I}{mhT^2} ,$$

when its amplitude of oscillation is small. Here I indicates the moment of inertia of the pendulum about its oscillation axis, m the mass of the pendulum, and h the distance between the oscillation axis and the centre of gravity of the pendulum.

Namely, g is inversely proportional to T^2, or gT^2 is constant at any point for a definite pendulum.

Using a pendulum, we determine its period of oscillation T_A at a base station A where the gravity g_A is known, and next we move the pendulum to another station B gravity of which is g_B and determine the period T_B there. Then we have the relation

$$\frac{g_B}{g_A} = \frac{T_A^2}{T^2_B}$$

from which g_B can be determined, with the values of T_A and T_B obtained from the measurements, provided g_A is known. This is the principle of the relative measurement of gravity by means of a pendulum.

There are two types of pendulum, thus, "Sterneck pendulum" and "bar pendulum" like the Gulf pendulum. The former, made of brass, has a weight at one end, and the latter is made of quartz in the case of the Gulf pendulum (Fig. 2).

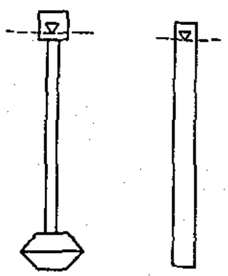


Fig. 2

Placing a knife-edge at an appropriate position of a pendulum, we can make its period of oscillation minimum. Then a small change of distance h between the knife-edge and the centre of gravity, for instance by the thermal expansion or contraction, has a minimum effect on the period. The pendulum which has such a character is called a minimum pendulum.

Formerly the Sterneck pendulum was often used, but nowadays the rod type has become common.

Differentiating the equation

$$gT^2 = \text{const},$$

we have $$\frac{dg}{g} = -2\frac{dT}{T}.$$

If the value of gravity is to be measured to the order of 1 mgal, the accuracy of the measurement should be

$$\frac{dg}{g} = \frac{1 \text{ mgal}}{980 \text{ gal}} = 10^{-6}$$

Therefore, the accuracy of measurement of period, dT/T, should be 5×10^{-7}.

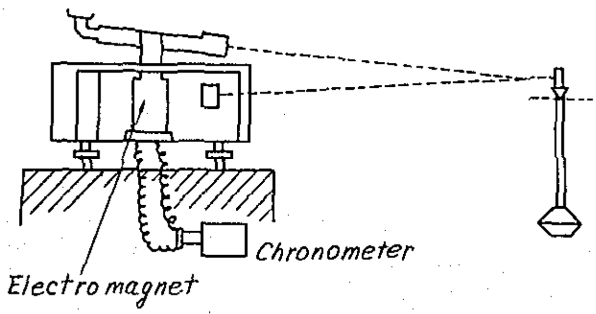


Fig. 3

This utmost high accuracy in the pendulum observation may be attained by the coincidence method.

The pendulum usually has a period of nearly 1 second. The slit of the coincidence apparatus opens exactly every one second by means of the electro-magnet and the break-circuit chronometer. The light from the slit is sent to the mirror of the oscillating pendulum, which reflects the light to the field of the telescope, where the image of the thin slit is focussed.

If now the image just coincides with the centre line of the filed, the image will move apart from the line with the following seconds, since the period of the pendulum is not exactly one second. The difference of time will increase with seconds, and at last it will attain one second after some seconds, say C sec. reckoned from the initial position of the image at the centre line; that is to say, we have again the coincidence of the image and the centre line. C is called the coincidence interval.

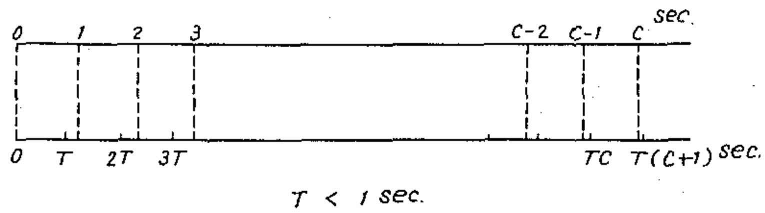


Fig. 4

When $T < 1$ sec, $C = T(C+1)$ or $T = 1 - \frac{1}{C+1}$

When $T > 1$ sec, $C = T(C-1)$ or $T = 1 + \frac{1}{C-1}$.

C may be determined as a mean value taken from a large number of oscillations. With the observed C, T can be calculated by the above formula.

In the case of observing C,

$$\frac{dT}{T} = \frac{1}{C+1}\,\frac{dC}{C} \qquad \text{when } T < 1 \text{ sec}$$

$$\frac{dT}{T} = \frac{-1}{C-1}\,\frac{dC}{C} \qquad \text{when } T > 1 \text{ sec.}$$

Considering only the accuracy of the measurement, we have

$$\frac{dT}{T} = \frac{1}{C}\,\frac{dC}{C}$$

In the Sterneck pendulum, C = 70 sec. approximately.
So when T is measured, dT/T should be less than 5×10^{-7}, while when C is measured, dC/C is measured, becomes 70 x dT/T, i.e. the measurement becomes much easier.

In the coincidence observation, the image from the minor of the pendulum crossed the centre line twice in opposite directions during an oscillation of the pendulum. The above C means the coincidence of the image and the centre line when the images crossing only in one direction are taken into consideration. If we consider the coincidence for both the images crossing in opposite directions, we have to take C' instead of C, where C' = 1/2 C. In this case,

When T 1 sec, $T = 1 - \frac{1}{2C+1}$ or $\frac{T}{2} = \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{C' + \frac{1}{2}}$ sec

When T 1 sec, $T = 1 + \frac{1}{2C-1}$ or $\frac{T}{2} = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{C' - \frac{1}{2}}$ sec.

Usually C' is measured.

F.A. Vening Meinesz devised a method eliminating horizontal accelerations of the support of pendulum. The soft land in Holland made him think of such a device.

Here used two similar half second pendulums which oscillate simultaneously in one vertical plane in opposite directions such other.

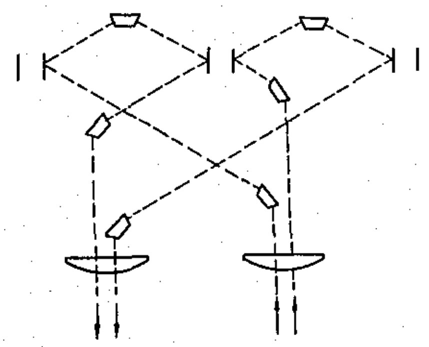


Fig. 5

In Fig. 5 m_1, m_2, m_3 are the pendulums. If we combine, for instance, m_1 and m_2, and send a light beam to the mirror of m_1, then the light is reflected by a prism to m_2, from where it goes to the coincidence apparatus. The light from the combination of two pendulums may be considered to come from a fictitious pendulum.

For a physical pendulum, the equation of motion is

$$\frac{d^2 \theta}{dt^2} + \frac{g}{\ell} \text{sing } \theta = 0 ,$$

If this pendulum is disturbed by a resultant of horizontal accelations in the direction of oscillation, the equation becomes

$$\frac{d^2 \theta}{dt^2} + \frac{g \theta}{\ell} - \frac{1}{\ell} \frac{d^2 f}{dt^2} = 0 ,$$

where $\frac{d^2 f}{dt^2}$ is the horizontal acceleration of the supporting point of the pendulum. The negative sign of $\frac{d^2 f}{dt^2}$ means that the direction of acceleration is the same as that of the motion of pendulum.

When the two pendulums are used side by side, we have

$$\frac{d^2 \theta_1}{dt^2} + \frac{g}{\ell_1} \theta_1 - \frac{1}{\ell_1} \frac{d^2 f}{dt^2} = 0 \qquad \text{for } m_1$$

$$\frac{d^2 \theta_2}{dt^2} + \frac{g}{\ell_2} \theta_2 - \frac{1}{\ell_2} \frac{d^2 f}{dt^2} = 0 \qquad \text{for } m_2$$

Taking the similar pendulums, i.e., $\ell_1 = \ell_2 = 1$, and combining the both, we have

$$\frac{d^2 (\theta_1 - \theta_2)}{dt^2} + \frac{g}{\ell} (\theta_1 - \theta_2) = 0 ,$$

which is the equation of motion for the free oscillation of fictitious pendulum having the same period as of m_1 and m_2 and the phase of $\theta_1 - \theta_2$. Therefore, by determining the coincidence interval of this fictitious pendulum, we can find the period of the original pendulum, disregarding $\frac{d^2 f}{dt^2}$.

Vening Meinesz used this method in the submarine, submerged below the depth of 30 m, and measured the gravity at sea. Since water oscillates as a surface wave, the effect of sea waves on the submarine may be almost negligible.

In 1923, as a first survey he made a trip to Java from Holland via Suez Canal. He found the gravity at sea is somewhat larger than the value on the land of the same latitude.

Nowadays the method of fictituous pendulum is used widely on land in order to eliminate the horizontal disturbance. It is not to mention that the oscillation of the two pendulums in the opposite directions may cancel a greater part of the effect of 'Mitschwingen' of the pendulum support.

The pendulum observation takes time, and a set of the instrument are bulky and heavy. Today gravimeters are almost exclusively used. Because they are portable and need only a few minutes for one station.

The gravimeter measures the difference of gravity at two points to 0.1 - 0.01 mgal.

There are some kinds of gravimeters. However, they are classified into two types - stable and unstable.

As a stable type we take the Askania's gravimeter.

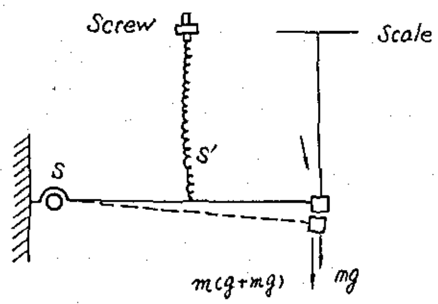


Fig. 6

In Fig. 6 if mg increases to m(g + mg), then the beam inclines from the original horizontal position. By the rotation of a screw D attached to the spring s', the beam is restored to the original position with the use of light and scale. The change of spring s' is proportional to the rotation, which is in turn proportional to mg.

The method of observation is called the "zero-method" or null method".

As an unstable type, the LaCoste Romberg gravimeter is dealt with here. This is very sensitive and mostly used for geodetic purposes as well as prospection.

A mass m is suspended by a spring as is shown in Fig. 7. If the balance is horizontal and in equilibrium, the moment by g is balanced with the moment due to the spring.

$$mg.l = kb \cos \psi .\ l = kal \qquad (1)$$

where kb is the force of the spring in the direction BC and changes with the length b, k being constant.

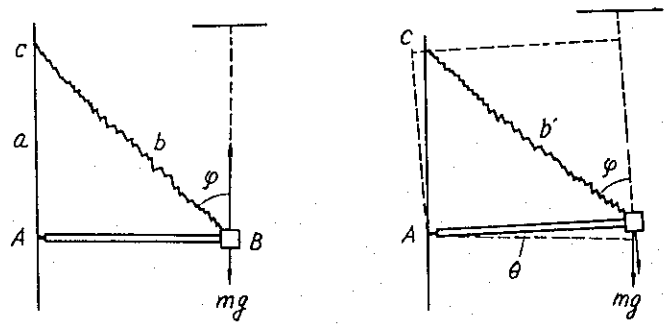


Fig. 7

When the force of the spring is proportional to its length, the spring is called a 'zero length spring'.

In the same gravity field, if the beam is not horizontal but makes an angle θ with the horizontal direction, b decreases to b'.
In this case,

The clockwise moment = mg cos θ.l

The counter-clockwise moment = kb' cos ψ'.l = k a cos θ.l

From (1) we find that both the above moments are equal. So the balance

in this case is also in equilibrium. That is to say, it is in an astatic condition and the equilibrium is maintained independent on g. This means the state is completely unstable. The mass m can take any position.
For instance, we imagine a door which can stop at any position. With a small force it can change the position a great deal. If the supporting vertical line is slightly inclined, the door becomes somewhat stable, but still sensitive to a small force.

In fact we can not attain the extreme condition: force of the spring = kb, but make the instrument very sensitive to a small change of gravity. When θ is small,

$$mg \cos \theta . 1 = ka \cos \theta . 1 + C\theta,$$

where C is a proportional constant.

Therefore, a small displacement $C\theta$ of the mass is proportional to $C\theta g$.

3. Method of survey

The gravimeter is so sensitive that a change of 0.01 mgal may be detected. Instead, the condition of the instrument varies considerable with time, probably due to the elastic creep of the spring. If we observe the gravity continuously at a point, the reading of the gravimeter change with time. The change is mostly due to what is called the 'drift'.

In the case of the gravity survey, the effect of the drift should be eliminated as far as possible. This will be accomplished by means of the method of looping. It is advisable to close a loop within a relatively short interval of time, say, two hours.

The change of readings of the two repeated measurements at a base point shows mostly the drift occurred during the measurements in the loop. This drift may be distributed to the points of the loop according as the order of observation.

For the pendulum observation, we do not need to take the drift in considerations. But its sensitivity is 0.2 - 0.25 mgal even in the recent well designed instrument.

If the gravity at one station is known, the values of gravity at other stations can be relatively determined by the pendulum observation or the gravimeter measurement, compared with that of the first a station.

The fundmanetal station of gravity is now in Potsdam, where the gravity was determined absolutely with the use of reversible pendulums by F. Kühnen and Ph. Furtwängler at nearly the beginning of the twentieth century:

Geodetic Institute, Potsdam

= 52° 22'. 86 N

= 13° 4'. 06 E

H = 87 m

Potsdam = 981.274 ± 0.003 gal.

The gravity value based on this Potsdam is called that of 'Potsdam: gravity *system*'.

From later absolute measurements at other places the value at Potsdam has been considered to have to be subtracted about 13 gal.

All the gravimeters are affected by the change of temperature. Some of them are made to compensate the effect. But most of them are kept at a constant temperature 40° or 50°C by batteries.

The readings of gravimeter are converted to the change of gravity Δg. That is, we must know the scale value of the instrument, which means how many change of gravity corresponds to one scale division.

We select two points where the relative values of gravity are known. For the points, we usually take them in a nearly N - S direction for getting a relatively large difference of gravity between the two points, or take two points, one of which is at the foot of a mountain and the other at a higher place. In the latter case we have to take care the difference of atmospheric effects at the two points. From the measurement at the two points, the scale value can be determined. The line connecting the two points is called the 'calibration line' for gravity. To minimize the drift effect, the shorter the time is, the better the result for the calibration is.

Concerning the spacing of observation points, it depends on the object of the survey. To detect relatively small ore bodies, the spacing is, say 20 m or 30 m. To find out major geological structures, a distance of 1 or 2 km or more is taken as the spacing.

When the survey area is large, several base points are selected and they are connected to one reference point.

As to the gravity points, we must know its exact position and its height. The position should be known not only for detecting underground bodies, but also for the latitude correction given to the observed value. The height is necessary for the elevation correction. Bench marks are often taken as observation points.

In order to determine the position and the height of the observation point as well as to calculate the terrain correction, preliminary topographic surveys are made with the use of transit and level. For height determination the barometer is sometimes used. It is preferable to measure the height at most to 3 cm, which corresponds to a difference of 0.01 mgal.

In gravity surveys in water, sometimes a water tight gravity meter is set on the bottom of the water. By remote control on a boat the meter is levelled with a motor drive device. The reading is given on a photographic plate. Sometimes an observer with a gravimeter enters a diving bell which is immersed to the bottom of the water.

Gravity measurement can also be done on a boat by the use of special device to eliminate the oscillation of the boat.

II. Reduction of observed gravity

1. Normal gravity

If we extend the mean sea level into the land through narrow channels, we have a closed level surface, which is called the 'geoid'.
Here, the channels are assumed to be so narrow that the original sea surface does not change and also assumed to have no capillary effect.

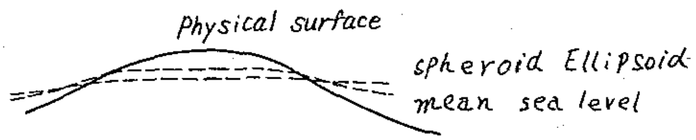


Fig. 1

The surface of the geoid is considered as a mathematical figure of the earth. However, it has an irregular form owing to the irregular mass distribution near the surface of the earth.

Instead of the geoid, a level surface which is very close to the geoid and has a rotation figure is considered. This idealized earth is called the 'earth spheroid' or 'niveau spheroid'.

The gravity on the spheroid is expressed by

$$\gamma_o = \gamma_e(1 + \beta\sin^2\phi - \beta' \sin^2 2\phi)$$

where γ_o = gravity at a latitude ϕ at sea level

γ_e = gravity at the equater at sea level

β, β' = constants depending on the data used for their determination

This is called a gravity formula, and γ_o calculated by the formula is called the normal gravity at a latitude ϕ.

The international formula 1930 is

$$\gamma_o = 978.049(1 + 0.005\ 288\ 4 \sin^2\phi - 0.000\ 005\ 9 \sin^2 2\phi)$$

which is widely used for calculating the normal gravity at a point at sea level.

To compare an observed value of gravity with the normal gravity, we usually give three reductions to the observed gravity, i.e. free-air reduction, terrain reduction and Bouguer reduction.

2. Free-air reduction

If the observation station is H m high above sea level, the observed g is smaller than that expected at sea level.

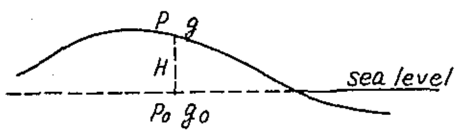


Fig. 2

Denoting g_o the gravity at P_o at sea level just below the observation station P,

$$g_o = G \frac{E}{R^2}$$

To reduce the observed gravity g to the value g_o at sea level, we have to add to g the following value:

$$\delta g_f = g_o - g = G \frac{E}{R^2} - G \frac{E}{(R+H)^2}$$

$$= G \frac{E}{R^2} \left(2\frac{H}{R} - 3 \frac{H^2}{R^2} + \ldots\ldots\right)$$

Sg_f is called the free-air reduction applied to g.

For the stations less than the height of 2000 m, the second term of the above equation is neglected. So

$$Sg_f = g_o - g = G \frac{E}{R^2} \times 2 \frac{H}{R} .$$

Taking the average value of gravity for GE/R^2 and the average radius of curvature for R, we have

$$Sg_f = 0.3086 \text{ H mgal}, \quad \text{H in m.}$$

When observation point is below sea level, H is negative and so has to be subtracted from g.

When H = 3 m, the value of reduction amounts to approximately 1 mgal. Today gravity can be measured to 0.01 mgal. For 0.01 mgal, the height of observation station should be known to the accuracy of 3 cm.

3. Terrain reduction

The mass above the observation station P attracts a unit mass at P upwards, and the mass defect below P gives upward attraction too. Therefore, if we substract the effects due to upper mass and lower negative mass from observed g, we have a value of gravity g' when P lies on a flat land having a height of H above sea level. This reduction is called the reduction g' is always larger than g.

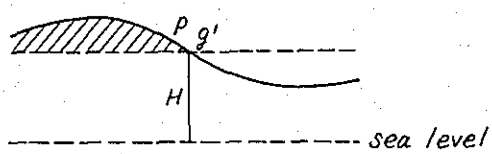


Fig. 3

The terrain reduction is calculated on the following basis.

We prepare a transparent sheet on which concentric circles and radial lines are drawn. The sheet is placed on a topographic map of the surveyed area, the gravity station being centred. The mean height h of each compartment of the sheet is read from the map.

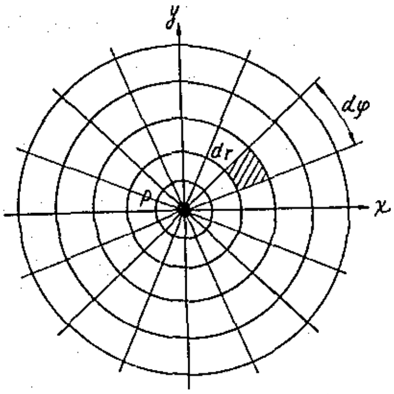


Fig. 4

Taking the sectoral prism of a compartment having a height h, we calculate its gravitational effect at P.

A small element dm of the prism is taken and the coordinate of its centre is represented with a cylindrical coordinate (r, ψ, z). The vertical attraction dg at P due to the element is

$$dg = G\rho \frac{rd\psi,\ dr,\ dz}{r^2 + z^2} \cdot \frac{z}{\sqrt{r^2 + z^2}}$$

The attraction of the prism is

$$\delta g = G\rho r d\psi dr \int_o^h \frac{z\ dz}{(r^2 + z^2)^{3/2}}$$

$$= G\rho\ (1 - \frac{r}{\sqrt{r^2 + h^2}})\ d\psi dr$$

$$= G\rho\ (1 - \cos\theta)\ d\psi dr$$

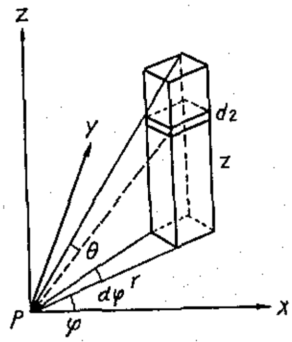


Fig. 5

The attraction depends only on the angle , which is the vertical angle between the vectors r and $\sqrt{r^2 + h^2}$.

We calculate all the effects due to the masses of compartments and add them. The effect is independent on the sign of θ. Because $\cos\theta = \cos(-\theta)$.

The terrain reduction $g' - g = Sg_t$ is always positive and has to be added to the observed g.

Since the terrain effect is generally small, it may be often neglected.

4. Bouguer reduction

g' gives the gravity at P which is located on an infinite slab having a thickness of H and lying horizontally between P and sea level.

If we subtract the effect of the mass of the slab on P, we have the gravity at P only due to the mass underneath sea level. The effect of the slab is given by $2\pi G\rho H$, which is a reduction to be subtracted from the observed g, and is called the Bouguer reduction, which is expressed by

The effect of the slab is derived from the last equation:

$$\delta g = G\rho \left(1 - \frac{r}{\sqrt{r^2 + h^2}}\right) d\psi dr.$$

Substitute H instead of h and integrate the equation from 0 to 2π and from 0 to ∞ with respect to ψ and r respectively.

$$\delta g_B = G\rho \int_o^{2\pi} d\psi \int_o^{\infty} dr \left(1 - \frac{r}{\sqrt{r^2 + h^2}}\right)$$

$$= 2\pi G\rho \left| r - \sqrt{r^2 + H^2} \right|_o^{\infty}$$

$$= 2\pi G\rho H .$$

Putting $G = 6.673 \times 10^{-8}$ and $\rho = 2.67$, which is the mean density of the granite consisting of the upper part of the crust,

$$Sg_B = 0.1119 \text{ H mgal}, \quad \text{H in m.}$$

For H = 9 m, Sg_B = 1 mgal.

Combined Sg_B with the free-air reduction, the gravity at sea level is obtained, and it is usually designated by g_o''.

$$g_o'' = g + \delta g_f - \delta g_B ,$$

where $Sg_f - Sg_B = (0.3086 - 0.1119)H = 0.1967\ H.$

$$g_o'' = g + 0.1967\ H, \quad H \text{ in m}, \ \rho = 2.67.$$

If the terrain reduction is considered,

$$g_o'' = g + Sg_f + Sg_t - Sg_B$$

The sign of the Bouguer reduction to be added to the observed gravity is always opposite to the free-air reduction.

5. Gravity anomalies

A reduction gravity value minus the normal value at the same latitude as the observation station is called the gravity anomaly.

$$g_o - \gamma_o = \Delta g_o \qquad \text{Free-air anomaly}$$

$$g_o'' - \gamma_o = \Delta g_o'' \qquad \text{Bouguer anomaly.}$$

The gravity anomaly is useful for finding out underground mass distribution. Which anomaly is used depends on the purpose.

The result of gravity observation for international network is usually tabulated as follows:

Sta.	ψ	λ	H	g	g_o	g_o''	$g_o - \gamma_o$	$g_o'' - \gamma_o$

The unit commonly used for each column is given below.

ψ :	0.1	or	" (second)
λ :	0.1	or	"
H :	0.1 m	or	0.01 m
g :	0.1 mgal	or	0.01 mgal

6 Reduction for prospecting

For local prospecting, the observed gravity is usually corrected for elevation, topography, and latitude. The words, correction and reduction can be used similarly.

(a) Elevation correction

A datum plane is chosen adequately for the correction instead of sea level. Usually a horizontal plane through the lowest station in the surveyed area is taken as the datum plane.

The correction Sg_e includes the free-air and the Bouguer correction and is given by

$$Sg_e = 0.3086\ h - 2\pi G\rho h$$
$$= 0.3086\ h - 0.041\ 91\,\rho\, h\ ,$$

where h is the height of observation station above the datum plane and ρ is the density taken for the slab between the station and the date plane.

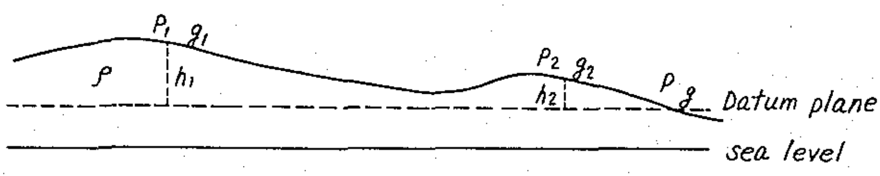


Fig. 6

(b) Terrain correction

Most of the effect due to topography of a large scale are eliminated by the Bouguer correction. So the correction required is that of the effect due to near terrain. As already stated, the upward attraction of hills higher than the station P and also the upward effect due to valleys have to be removed, so as to have the gravity at P lying on an infinite slab of thickness h reckoned from the datum plane.

Both the calculated effects are to be added to g and the corrected g is always increased.

For the correction, a method has been previously given. However, the Hammer's method is often used. His transparent chart, having concentric circles and radial segments is superimposed on a topographic map around the gravity station P. Take the difference between the mean height of a compartment of the chart and the height of the gravity station. The sign of the height difference is not taken into account. The correction for every compartment is obtained from the Hammer's Table if the height difference and density ρ are known, and all the corrections thus obtained are added to g.

$g + Sg_e + Sg_t$ gives the gravity value on the datum plane when the masses above the station and those of negative density below the station are removed.

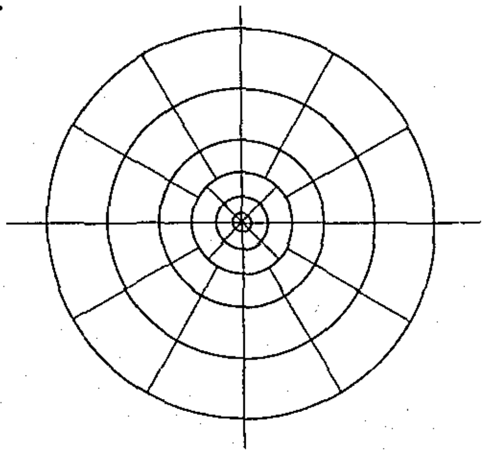


Fig. 7

(c) Latitude correction

The change of gravity with latitude ψ, Sg_ℓ, is given by differentiating the gravity formula with respect to ψ.

$$\gamma o = 978.049(1 + 0.005\ 288\ 4 \sin^2\psi - 0.000\ 005\ 9 \sin^2 2\psi)$$

$$\frac{d\gamma o}{d\psi} = 5.172 \sin 2\psi$$

$$\text{or } \delta g_\ell = 5.172 \sin 2\psi \cdot d\psi \text{ gal}, \quad d\psi \text{ in radian}$$

If s is taken as the distance on the earth's surface in a N - S direction,

$$s = R\, d\psi$$

$$Sg_1 = \frac{9.172}{6370} \sin 2\psi \cdot s, \quad \text{gal, s in km.}$$

$$= 0.812 \sin 2\psi \cdot s, \quad \text{mgal,}$$

which expresses the change of the normal gravity when the station at ψ moves to $\psi + \delta\psi$ or moves northwards by a distance s km.

When a standard station, which has approximately the average latitude of the surveyed area, is selected, the corrections to the other stations are calculated only by knowing the distances in the N-S direction from the standard station. For northern points Sg_e has to be subtracted and for southern points vice versa.

The corrected gravity at $P = g + Sg_e + Sg_t - Sg_1$,
which gives a kind of Bouguer anomaly, and is employed as the basis of the calculation for prospecting.

When the surveyed area is large, Sg_1 for a station having latitude $\psi + \delta\psi$ is calculated from the difference of two normal gravity:

$$Sg_1 = \gamma o\ (\psi + \delta\psi) - \gamma o\ (\psi),$$

where ψ is the latitude of the standard station.

(d) Earth-tide correction and drift correction

The solid earth changes its form with the relative position of the moon and the sun. S_o the gravity changes.

This earth-tide correction is made by taking the record of the change of gravity due to earth-tide during surveys or by using tables especially published for this purpose. However, if we repeat the gravity measurement at the base station in a short interval, the tidal effects at the other stations taken between the two repeated measurements at the base will be mostly eliminated together with the instrumental drift, provided the effect being assumed to have changed linearly.

As to the drift, it has been stated previously.

7. Results of measurements

Values of gravity anomaly are plotted on a map and isogal lines of Bouguer anomaly or free-air anomaly are drawn with a suitable interval.

From these gravity maps, conclusions about underground structures are drawn always in consistence with geology and other facts.

In Fig. 8, a few examples are illustratively given, concerning the relation of underground structure and gravity profile taken on the ground through the point of the maximum effect above the source of the anomaly.

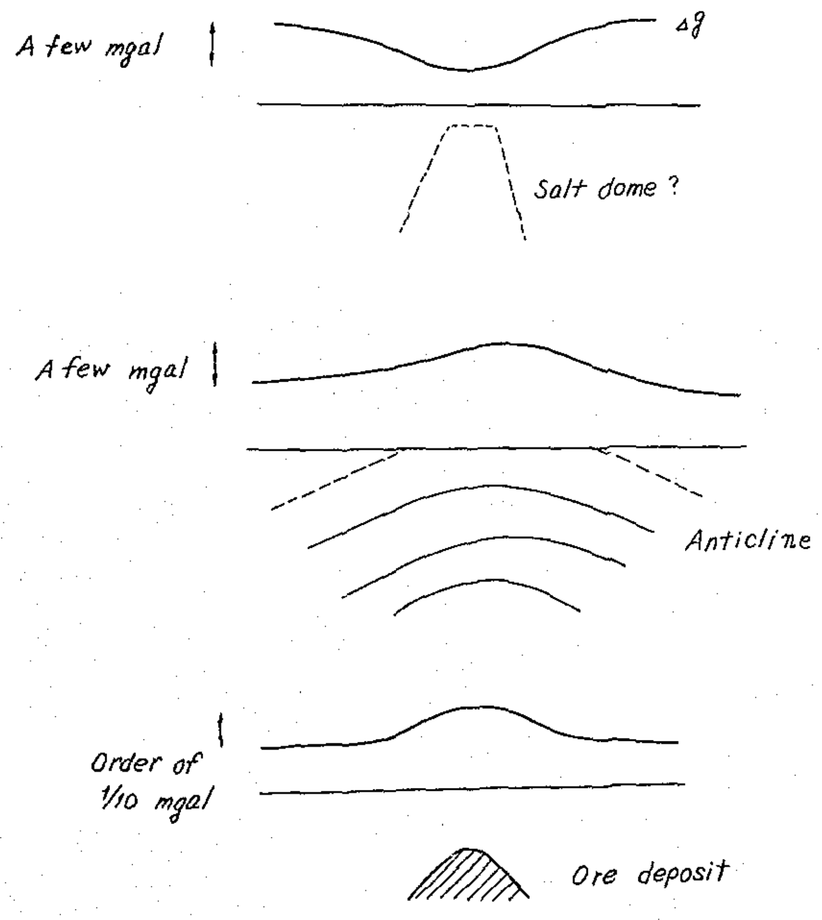


Fig. 8

III. Theory of isostasy

1. Isostasy

Isostasy is one of the characteristic states of the earth's crust. The crust has always a tendency to keep this equilibrium state, and if there occurs a certain change, for instance, due to the deposition of sediments or the denudation of rocks by erosion, the crust tends to restore the original equilibrium state. But as the crust has a property of rigidity, it can not follow the change of surface mass distribution at once. Even if the crust can restore the state, it takes time to bring back to the isostatic state. When the surface change is limited to a small locality, the restoration may not be fulfilled on account of the rigidity of the crust. When the area of the change is large enough, the restoration will be accomplished after some years. The limit of the area subject to restoration is called the "area of compensation". The area is believed in general to be a circle having a radius of 60 ~ 150 km.

A piece of jelly is strong to retain its form, in other words, it has a property of rigidity. However, a large volume of jelly can not retain its form like a viscous fluid, that is to say, jelly has a plastic property when it has a large size. The crust may have the same property as jelly. So a large portion of the crust can behave just like a viscous fluid. Hence the crust is considered to have as a whole a tendency to follow the isostatic adjustment.

2. Pratt Theory and Airy Theory

According to the Pratt idea stated above, the depth of compensation is taken at the depth where the level surface and the surface of equal pressure coincide and its depth has been thought about 100 km. Above this depth of compensation, the surface of equal pressure approaches to the physical surface of topography and differs more and more from the level surface (Fig. 2).

Take pillars of different density ρ and of the different height h which is inversely proportional to the density such as

$$\rho_1 h_1 = \rho_2 h_2 \quad .....,$$

and let them float on a liquid of a denser density ρ_o. They will rest with their bottoms at the same depth which corresponds to the depth of compensation. This example shows the Pratt theory (Fig. 4).

From the observations of the plumb line deflections in India, G.B. Airy (1801 - 1892) also thought of the idea of isostasy almost at the same time as Pratt. He thought that continents having a constant density float at the surface of the substratum having a larger density just like icebergs at the ocean surface.

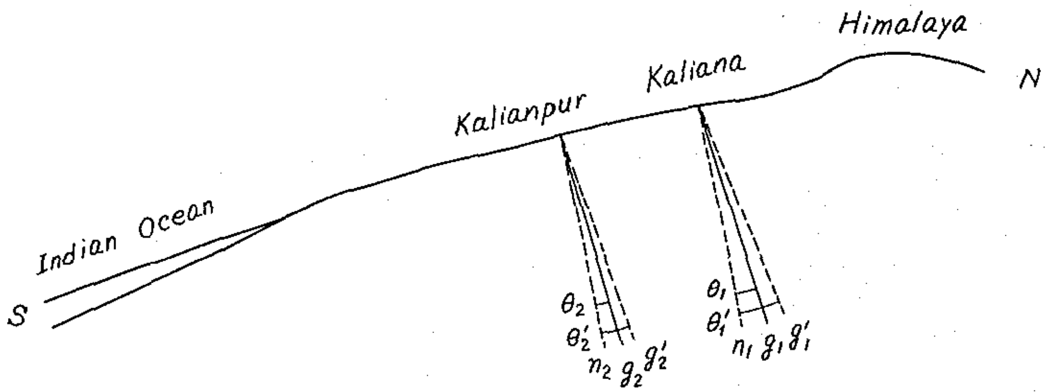
If we take pillars of the same density and of different lengths and dip them into a liquid having a larger density, they will float as Fig. 5 shows. For the Airy theory the relation between continents and the substratum is given by this illustration.

If we consider a level passing the bottom of the longest pillar, the level has the similar property as the depth of compensation, on which the masses of pillars cause the same pressure.

Suppose the earth's crust is in the state of isostasy. Then, following the deposition or denudation of rocks, the crust of that part must move to restore the original equilibrium state by sinking into the substratum or by getting substances from the substratum. As a matter of fact it is well known that the delta area of the mouth of a large river is subsiding owing to an immense deposit of debris transported by the river. On the other hand, it is also believed by geologists that the denudated area is being uplifted.

These geological phenomena can be explained by both the theories of Pratt and Airy. But most of the geological layers are spread horizontally and they have usually almost the same density along the horizontal direction.

From this point of view, the Airy's theory seems to be closer to the geological fact than the Pratt's. But the result of isostatic calculation made by Hayford and Bowie (described later) shows that the Pratt theory on which their calculation was based is to be accepted as elucidating a general state of the earth's crust. In fact, both the theories have basically the same idea that the earth's crust floats on the substratum. The idea itself is the basis of isostasy.



n_1, n_2	:	Geodetic normal to the ellipsoid.
g_1, g_2	:	Astronomical normal to the equipotential surface; i.e., direction of gravity.
θ_1, θ_2	:	Plumb line deflection actually measured.
g_1', g_2'	:	Calculated normal to the equipotential surface, the attraction of the Himalayas being considered.
θ_1', θ_2'	:	Plumb line deflection calculated for the effect of the Himalayas .

Pratt found $(\theta_1 - \theta_2) < (\theta_1' - \theta_2')$

Fig. 1

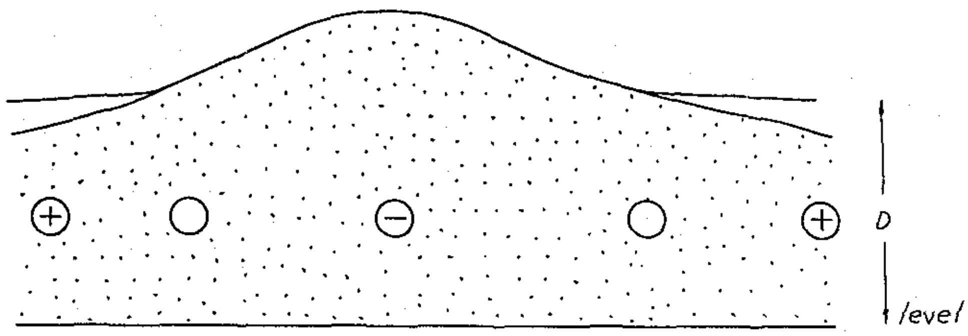

D
level

Fig. 2

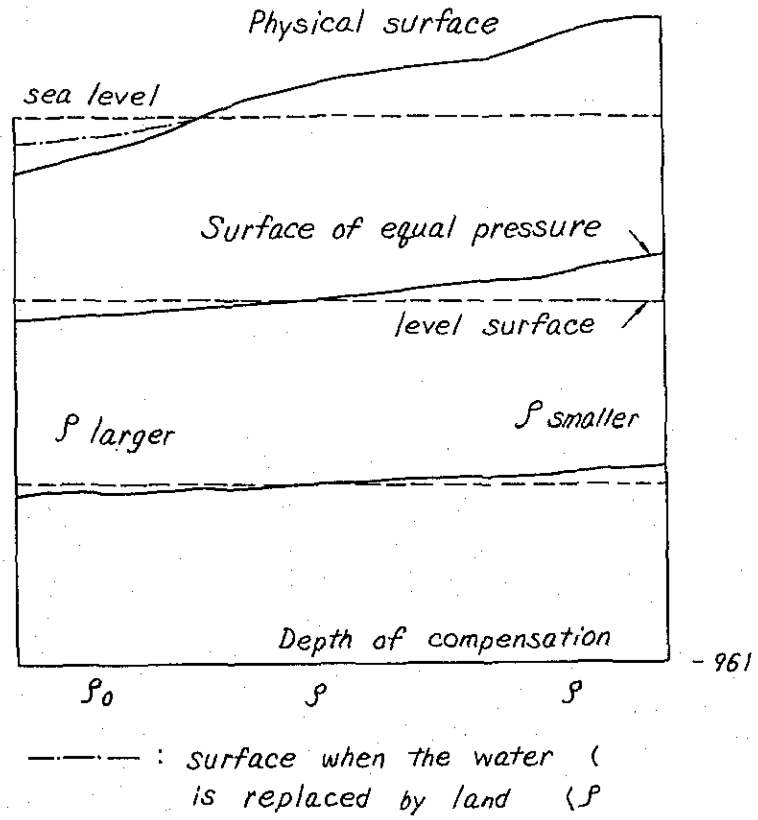
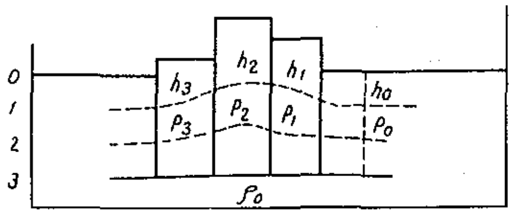

Physical surface
sea level
Surface of equal pressure
level surface
ρ larger
ρ smaller
Depth of compensation
- 961
ρ0
ρ
ρ
—·— : surface when the water (
is replaced by land (ρ

Fig. 3



1, 2 : surface of equal pressure
3 : surface of equal pressure and also level surface
$\rho_n h_n = \rho_0 h_0$

Fig. 4

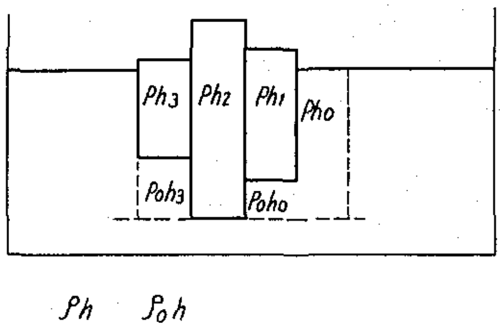


Fig. 5

3. Isostatic reduction - Pratt-Hayford Method

In the early part of the twentieth century J.F. Hayford (1868 - 1925) calculated the isostatic effect on the plumb line deflection measured in the United States. Although his calculation was based on the Pratt idea, the treatment of isostasy was a little different from the Pratt's original idea. The depth of compensation D was measured from the physical surface in order to simplify the calculation. Therefore the vertical distance of the layer of compensation from sea level depends on the height of topography (Fig. 6).

If a layer of equal pressure is taken at the depth D from sea level just as the Pratt idea, the following equation holds (Fig. 6):

$$\rho D = (\rho - \Delta\rho)\, D + \rho h \qquad (1)$$

$$\Delta\rho = \frac{h}{D}\,\rho\ ,$$

where ρ is the density of a rock column of length D, of which the top surface is the coastal plain and $\Delta\rho$ is a compensating density of a column, of which the upper part forms a mountain of height h. Following to the Hayford's method, W. Bowie calculated the isostatic effect on the gravity observed in the United States.

The isostatic reduction for the observed gravity has two procedures, i.e., the topographic effect on gravity due to the mass above sea level is removed and the effect of the mass deficiency, which is considered to extend from the physical surface through the distance of D, is eliminated. The latter is the compensating effect.

The masses to be considered around a gravity station are divided into sectoral pillars by concentric circles and radial segments, and the gravity effect of each pillar is calculated similarly as the topographic correction (Fig. 7).

The gravity effect G_g of a cylinder at a point P' on its axis is given (Fig. 7)

$$\delta g_{cyl} = 2\pi G\rho(h + \sqrt{r^2 + H^2} - \sqrt{r^2 + (h + h')^2}) , \qquad (1)$$

where r and h are respectively the radius and the height of the cylinder, and h' is the distance from the attracted point P' to the upper surface of the cylinder.

The effect due to a cylindrical ring at a point on its axis is

$$\delta g_{cyl\ ring} = 2\pi G\rho(\sqrt{r_2^2 + h'^2} - \sqrt{r_2^2 + (h + h')^2} - \sqrt{r_1^2 + h'^2} + \sqrt{r_1^2 + (h + h')^2}) , \qquad (3)$$

where r_2 and r_1 are respectively the outer and inner radii of the cylindrical ring (Fig. 7). Based on this formula, the isostatic reduction is calculated.

For the topographic effect, Sg_t(ring), above sea level,

$$\delta g_t(\text{ring}) = 2\pi G\rho(\sqrt{r_2^2 + h'^2} - \sqrt{r_2^2 + (h + h')^2} - \sqrt{r_1^2 + h'^2} + \sqrt{r_1^2 + (h + h')^2}), \qquad (4)$$

where h is the height of the topography above sea level.

For the compensation effect δg_c(ring) due to the negative mass of density, $-\Delta P$ or -h/D.P

$$\delta g_c(\text{ring}) = -2\pi G \frac{h}{D} \rho\left(\sqrt{r_2{}^2 + h'^2} - \sqrt{r_2^2 + (D+h')^2}\right.$$
$$\left. - \sqrt{r_1^2 + h'^2} + \sqrt{r_1^2 + (D+h')^2}\right) \quad (5)$$

For the isostatic reduction, both (4) and (5) are subtracted from the observed gravity.

As h' is usually very small compared with D or r_1 (4) and (5) can be expanded as follows:

$$\delta g_{t(\text{ring})} = 2\pi G\rho \{r_2 - \sqrt{r_2^2 + h^2} - r_1 + \sqrt{r_1^2 + h^2} + F_1(h', h, r)\} \quad (6)$$

$$\delta g_{c(\text{ring})} = -2\pi G\rho \frac{h}{D} \{r_2 - \sqrt{r_2^2 + D^2} - r_1 + \sqrt{r_1^2 + D^2} + F_2(h', D, r)\}, \quad (7)$$

where the two functions $F_1(h', h, r)$ and $F_2(h', D, r)$ have a similar form and can be regarded as correctional terms, since their values are very small compared with the former four terms.

The cylinderical ring is divided into n equal sectoral pillars (Fig. 7), and the mean height h of each pillar above sea level is estimated from a topographical map. For such a sectoral pillar,

$$\delta g_t = \frac{2\pi G\rho}{n} (r_2 - \sqrt{r_2^2 + h^2} - r_1 + \sqrt{r_1^2 + h^2}) + f_1 \quad (8)$$

$$\delta g_c = -\frac{2\pi G\rho}{n} \frac{h}{D} (r_2 - \sqrt{r_2^2 + D^2} - r_1 + \sqrt{r_1^2 + D^2}) + f_2 \quad (9)$$

Here f_1 and f_2 are the corrections for h', and their values can be obtained from the tables prepared as the corrections which depend on the variables h' and h. When $h' \ll r_1$ & r_2, they are neglected.

Hayford divided the area around a station as is shown in Table 1.

The gravity effect of each compartment can be calculated by (8) and (9) with known r_1, r_2 and ρ, if h is estimated. ρ was taken 2.67 by Hayford. n pairs of gravity effect calculated by (8) and (9) for all compartments are summed up, and the sum is subtracted from the observed gravity at P'. Subtracting the sum from the observed value is called the isostatic reduction or correction.

For the ocean with depth h (Fig. 6),

$$\rho D = 1.027h + \rho(D-h) + \Delta\rho . D ,$$

where 1.027 is the density of sea water and $\Delta\rho . D$ is the compensating mass against the mass deficiency of the ocean, the density excess being considered from the ocean surface to the depth of compensation.

$$\Delta\rho = 1.643 \frac{h}{D}$$

The formulas for the calculation of the effect due to the mass deficiency of sea water and the effect due to the compensating excess mass extending from sea level to the depth of D are given:

$$\delta g_t = -\frac{2\pi G}{n} 1.643(r_2 - \sqrt{r_2^2 + h^2} - r_1 + \sqrt{r_1^2 + h^2}) + f_1' \quad (10)$$

$$\delta g_c = \frac{2\pi G}{n} 1.643 \frac{h}{D} (r_2 - \sqrt{r_2^2 + D^2} - r_1 + \sqrt{r_1^2 + D^2}) + f_2' , \quad (11)$$

where h' in f_1' and f_2' are the height of the observation station above sea level. When $h' \ll r_1$ & r_2, f_1 and f_2 are neglected.

The value of the depth of compensation is to be determined so that the isostatic anomalies at many gravity stations become the smallest. Hayford took 113.7 km for D, and Bowie 96 km.

The effect of topography and the effect of compensation have different signs. So the sum of these two effects gives a smaller value than the larger one of the two, but does not vanish at most cases even if the disturbing masses are far away from the attracted point. Therefore the isostatic reduction is extended all over the world.

The correction for the zones from A to L are calculated from the above formulas under the assumption that the sea level is horizontal.

M, N and O zones are calculated first with the above plane formulas and then correction is added with the use of the tables prepared by Hayford.

Zones farther than O zone are numbered 18, 17, etc. and calculated with spherical formulae, which are derived as follows:

Consider first the gravity effect dg at P' due to an elementary mass dm at P which is located above the niveau surface passing P' (Fig. 8).

Let d be the distance from P' to P and β the angle from horizontal plane through P' to the line P'P.

$$dg = G \frac{dm}{d^2} \sin \beta \tag{12}$$

Table 1 Zones and Compartments by Hayford

Zone	Outer Raduis	No. of Compartments
A	2 m	1
B	68	4
C	230	4
D	590	6
E	1,280	8
F	2,200	10
G	3,520	12
H	5,240	16
I	8,440	20
J	12,400	16
K	18,800	20
L	28,800	24
M	58,800	14
N	99,000	16
O	166,700 (=1°29'58")	28
18	1°41'13"	1
17	1 54 52	1
16	2 11 53	1
15	2 33 46	1
14	3 03 05	1
13	4 19 13	16
12	5 46 34	10
11	7 51 30	8
10	10 44	6
9	14 09	4
8	20 41	4
7	26 41	2
6	35 58	18
5	51 04	16
4	72 13	12
3	105 48	10
2	150 56	6
1	180 00	1

Denoting α the angle between OP' and OP, a the chord subtended by α, h the height of dm from the niveau surface P_o, R the earth's radius, then

$$d^2 = a^2 + h^2 + 2ah \sin\frac{\alpha}{2} ,$$

$$\because \cos\{\pi - \frac{1}{2}(\pi - \alpha)\} = -\sin\frac{\alpha}{2}$$

On the other hand,

$$\sin\beta = \frac{1}{d}\{R - (R+h)\cos\alpha\}$$

$$= \frac{1}{d}(\frac{a^2}{2R} - h\cos\alpha) ,$$

where the relation $R - R\cos\alpha = a^2/2R$ can be derived as follows (Fig. 9):

Let $P'A \perp OP_o$. Then $P'P_o$ becomes a tangent at P' to the circle passing through the three points P', A and B.

$$P_oP'^2 = P_oA \cdot P_oB$$

$$a^2 = (R - R\cos\alpha) \cdot 2R$$

From (12)

$$dg = G\frac{\frac{a^2}{2R} + h\cos\alpha}{(a^2 + h^2 + 2ah\sin\frac{\alpha}{2})^{3/2}} dm , \qquad (13)$$

which can be calculated with known α and h.

For an elementary mass dm at a depth h below the niveau surface, we have to use $-h$ instead of h in (13).

$$dg = G\frac{\frac{a^2}{2R} + h\cos\alpha}{(a^2 + h^2 - 2ah\sin\frac{\alpha}{2})^{3/2}} dm \qquad (14)$$

From (13) and (14) the topographic effect above sea level and the corresponding compensating effect of each compartment can be calculated, provided that the height of topography, or the depth of ocean is known.

Hayford prepared many tables to facilitate the calculations for his isostatic reduction. The isostatic reduction includes terrain and Bouguer reductions. So the gravity used for the isostatic anomaly, gi, is given as

$$g_i = g + \delta g_f - \delta g_i$$

where g is the observed gravity, δg_f the free-air reduction and δg_i is the sum of topographic and compensating effects, the latter two effects having different signs each other.

The isostatic anomaly, Δg_i, is

$$\Delta g_i = g_i - \gamma_o \qquad (16)$$

where γ_o is the normal gravity at sea level for the point under discussion.

4. Establishment of the Theory of Isostasy

The determination of the form of the earth is based on the measurements of arc length between two points on a meridian and the difference of their latitudes.

The measurement of latitude depends on the direction of gravity. Therefore the result of observation has to be usually corrected for neighbouring anomalous mass.

According to the theory of isostasy, it is not sufficient to remove only the effect of the mass above sea level. In addition to this elimination, the compensating negative mass in the underground has to be subtracted. If the isostatic compensation is perfect, the observed plumb line deflection may become zero by this reduction.

By the isostatic reduction applied for the observed gravity value, the gravity value may approach to that on an ideal earth, and the gravity anomaly approaches to zero.

Hayford calculated the isostatic reduction by his method, as stated in Section 3, for all the values of plumb line deflection observed in U.S.A., and found that the corrected plumb line deflection become to be about 1/10 of the observed deflection. This fact suggests that the crust has a property of isostasy, and by his calculations the thoery of isostasy has become to be recognized.

W. Bowie applied the isostatic reduction to observed gravity values. The result showed also that the gravity anomalies decreased to much smaller values.

The Hayford ellipsoid (1909) was determined from the arc measurements carried out only in the U.S.A., nontheless it has been taken as the basis of the international ellipsoid (1924). The reasons is based on the fact that the values of latitude used for the calculation were corrected for the isostatic effects.

Hayford ellipsoid 1909	International ellipsoid 1924
a = 6378 388 m	a = 6378 388 m
b = 6356 909	b = 6356 912
$f = \dfrac{1}{296.96}$	$f = \dfrac{1}{297.00}$

Isostasy is one of the most important principles in earth science, and as stated above, it has been established by the American geophysicists.

5. Isostatic Reduction - Airy-Heiskanen Method

The Airy-Heiskanen method of isostatic reduction is based on the Airy idea. Let the average crustal thickness be indicated by T. Then the mass of height h above sea level is compensated by the root of continent of depth t beneath the lower surface of the average crust, and the mass deficiency of ocean of depth h' is compensated by the anti-root of height t' above the lower surface of the average crust (Fig. 10).

If the density ρ of the crust is taken as 2.67 and that of the substratum ρ' 3.27. The thickness to of the root is given as

$$h\rho = t\Delta\rho \quad , \qquad \text{where} \quad \Delta\rho = \rho' - \rho. \qquad (17)$$

$$\therefore \ t = \frac{\rho}{\Delta\rho}\ h = 4.45\ h$$

The total crustal thickness T_c under continent is

$$T_c = T + h + t = T + 5.45\,h \qquad (18)$$

In the ocean, a column of mass deficiency is compensated by a column of mass excess in the anti-root.

$$(\rho - 1.03)\,h' = t'\Delta\rho \qquad (19)$$

$$t' = \frac{\rho - 1.03}{\Delta\rho}\ h' = 2.73\,h'$$

The total crustal thickness under sea is:

$$T_s = T - h' - t' = T - 3.73\,h' \qquad (20)$$

When the crust is isostatically compensated, its thickness is simply calculated from (18) or (20), where the normal thickness T is usually taken as 30 km.

The compensating effect of a ring of root of continent, and that of a ring of anti-root of ocean are given respectively just as formula (5) (c.f. Fig. 7):

$$\delta g_{c(ring)c} = -2\pi G\Delta\rho\{\sqrt{r_2^2 + (T+H)^2} - \sqrt{r^2 + (T+H+t)^2} - \sqrt{r_1^2 + (T+H)^2} + \sqrt{r_1^2 + (T+H+t)^2}\} \qquad (21)$$

$$\delta g_{c(ring)s} = 2\pi G\Delta\rho\{\sqrt{r_2^2 + (T+H-t')^2} - \sqrt{r_2^2 + (T+H)^2} - \sqrt{r_1^2 + (T+H-t')^2} + \sqrt{r_1^2 + (T+H)^2}\}, \qquad (22)$$

where H is the height of the gravity station above sea level, and t and t' are obtained from (17) and (19).

For the calculation of the isostatic reduction, the Hayford's zones and compartments are used. From A to O of the zones, the effects are calculated by the above plane formulae with the use of the tables prepared by Heiskanen. The tables are based for a point at sea level i.e. having $H \approx 0$. So the basic formulae used for the tables are those of (21) and (22) in which let $H = 0$.

The formula for a cylindrical ring just as (21) or (22) is used for the calculation of topographic effect of continent and ocean, which are respectively.

$$\delta g_{t(ring)c} = 2\pi G\rho\{\sqrt{r_2^2 + (H-h)^2} - \sqrt{r_2^2 + H^2} - \sqrt{r_1^2 + (H-h)^2} + \sqrt{r_1^2 + H^2}\} \tag{23}$$

$$\delta g_{t(ring)s} = -2\pi G(P - 1.03)\{\sqrt{r_2^2 + H^2} - \sqrt{r_2^2 + (H+h')^2} - \sqrt{r_1^2 + H^2} + \sqrt{r_1^2 + (H+h')^2}\} \tag{24}$$

The values corresponding to these plane formulae have already been tabulated by Hayford in his isostatic reduction.

For the zones from 18 to 1, the spherical formulae (13) and (14) are used for both the compensating and topographic effects as in the case of the calculation by Hayford.

The isostatic anomaly is similarly given by (15) and (16).

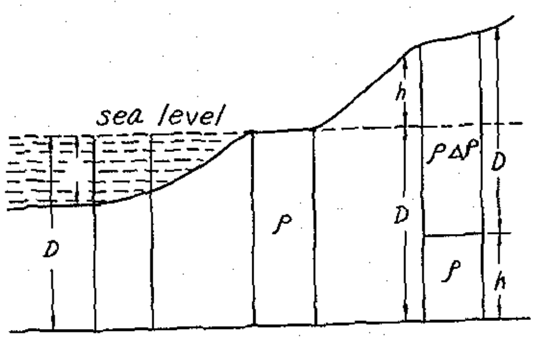


Fig. 6

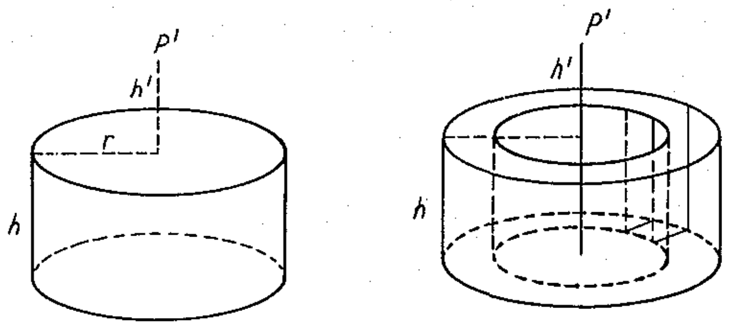


Fig. 7

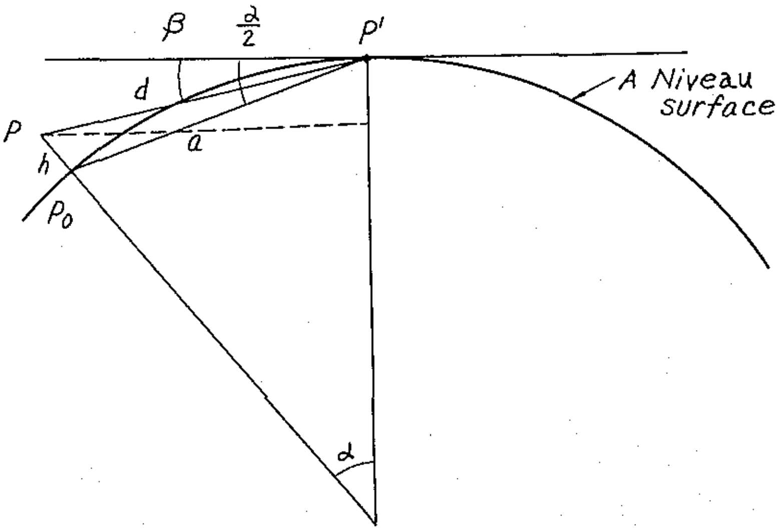


Fig. 8

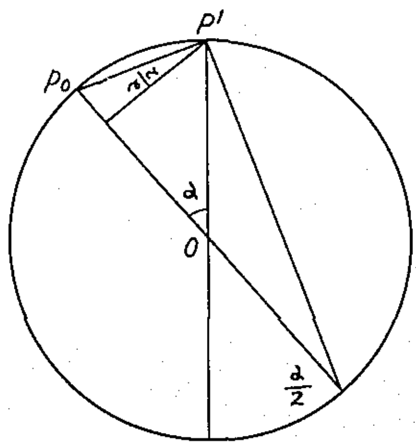


Fig. 9

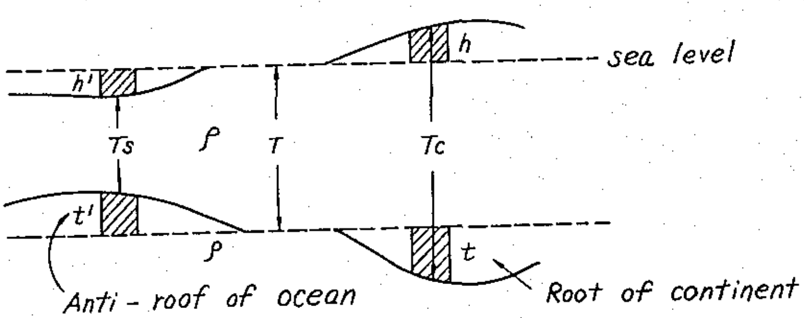


Fig. 10

IV. Gravitational attractions of some bodies having simple forms

To calculate the attraction of a body on a point, we usually first calculate the potential of the body at the point and then find the attraction by differentiating the potential with respect to the direction in which the attraction is required.

The potential is a function whose first derivative with respect to a direction gives the force along the direction.

1. Homogeneous spherical shell

In general, the potential of attraction, V, is given by the integration

$$V = G \int \frac{dm}{e} , \qquad (1)$$

where G denotes the universal constant of gravitation, which is according to the determination by P.R. Heyl,

$$G = 6.670 \times 10^{-8}\ cm^3/gr.\ sec^2 \quad (1930) \qquad (2)$$

$$= 6.673 \times 10^{-8} \quad " \quad " \quad (1942)$$

and $$m^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 , \qquad (3)$$

(x', y', z') being the coordinates of the point P where the attraction of the body is to be found, and (x, y, z) the coordinates of a point P having an elementary mass dm of the body (Fig. 1).

We use polar coordinates. The

to polar ones (r', Q', ψ') by the relations.

$$x' = r' \sin\theta' \cos\psi'$$

$$y' = r' \sin\theta' \sin\psi'$$

$$z' = r' \cos\theta'$$

If (x, y, z) are expressed in (r, θ, ψ) of polar coordinates

$$dm = \rho \cdot r d\theta \cdot r \sin\theta \, d\psi \cdot dr \qquad (4)$$
$$= \rho r^2 \sin\theta \;\; d\theta \;\; d\psi \;\; dr \; ,$$

where ρ denotes the density of the attracting mass (Fig. 1).

Let $\gamma = \;\; r, \; r'$ then

$$V = G \iiint \frac{r^2 \sin\theta \; d\theta \; d\psi \; dr}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} \qquad (5)$$

(a) Spherical Shell with Infinitely Small Thickness

Now we take z' - axis through p' (Fig. 2). Generality is kept, and γ becomes θ.

First the potential V due to a spherical shell with an infinitely small thickness dr is required.

$$V = G\rho r^2 dr \int_{\theta=o}^{\pi} \int_{\psi=o}^{2\pi} \frac{\sin\theta \; d\theta \; d\psi}{\sqrt{r^2 + r'^2 - 2rr' \; \cos \theta}}$$
$$\qquad (6)$$
$$= G\rho r^2 dr \cdot 2\pi \int_{o}^{\pi} \frac{\sin\theta \; d\theta}{\sqrt{r^2 + r'^2 - 2rr' \; \cos \theta}}$$

Put $\cos \theta = t$, then $\sin\theta \; d\theta = -d(\cos \theta) = -dt$, and the limits of integration: $\theta = 0 \rightarrow \pi$ becomes $t = 1 \rightarrow -1$. So (6) can be written

$$V = 2\pi G\rho r^2 \; dr \int_{1}^{-1} \frac{-dt}{\sqrt{r^2 + r'^2 - 2rr't}}$$

Since $\displaystyle\int \frac{dX}{\sqrt{a + bX}} = \frac{2\sqrt{a + bX}}{b}$,

$$V = 2\pi G\rho \; r^2 \; dr \left| \frac{-\sqrt{r^2 + r'^2 - 2rr't}}{rr'} \right|_{-1}^{1}$$

$$= 2\pi G\rho \; r^2 \; dr \; \frac{\sqrt{(r' + r)^2} - \sqrt{(r' - r)^2}}{rr'} \; ,$$

where $\sqrt{(r' - r)^2}$ has one of two values $\pm (r' - r)$ for $r' > r$ or $r' < r$ respectively.

(i) When $r' > r$, or p' is outside the shell, the potential becomes

$$V = 2\pi G\rho\ r^2 dr \cdot \frac{2}{r'} = G\frac{m}{r'}\ , \tag{7}$$

where m is the mass of the shell, i.e., $m = 4\pi r^2 \rho \cdot dr$.

This is the potential due to the whole mass concentrated at the centre of the spherical shell with a thickness dr. The force F is in the direction to the centre is

$$F = \frac{\partial v}{\partial r'} = -G\frac{m}{r'^2}\ , \tag{8}$$

where the negative sign means that the attractive force is direction opposite to that of increasing r'. In order to have the force to be positive, we will take sometimes a negative derivative of potential as the force.

(ii) When $r' < r$, or p' is inside the shell, the potential becomes

$$V = 2\pi G\rho\ r^2 dr \cdot \frac{2}{r} = G\frac{m}{r} \tag{9}$$

This depends on r and independent on r'. Therefore the potential inside a shall is everywhere constant.

$$-\frac{\partial v}{\partial r'} = 0\ , \tag{10}$$

i.e., there is no force inside a spherical shell.

(iii) When $r' = r$, (7) and (9) are equal. This means that the potential is continuous at the boundary. So, we can calculate the force on the surface from (8).

(b) Spherical Shell with Finite Thickness

Next we deal with the potential of a spherical shell with a finite thickness and a constant density.

Let r_1 and r_2 be the outer and the inner radii of the shell and r' the distance of an attracted point from the centre.

(i) Potential at an outside point p', r' > r.

From (7)

$$V = \int_{r2}^{r1} 4\pi G\rho r^2 dr \frac{1}{r'}$$

$$= 4\pi G\rho \left| \frac{r^3}{3} \right|_{r_2}^{r1} \quad \frac{1}{r'} , \tag{11}$$

$$= \frac{4}{3} \pi G\rho \ (r_1{}^3 - r_2{}^3) \ \frac{1}{r'} \quad ,$$

$$V = G \frac{\text{Total mass}}{r'}$$

$$- \frac{\partial V}{\partial r'} = G \frac{\text{Total mass}}{r'^2} \tag{12}$$

Thus the force is the same as that due to the whole mass concentrated to the centre.

(ii) Potential at an inside point, r' < r.

From (9)

$$V = \int_{r_2}^{r_1} 4\pi G\rho \ r \, dr$$

$$= 4\pi G\rho \left| \frac{r^2}{2} \right|_{r_2}^{r_1}$$

$$= 2\pi G\rho \ (r_1^2 - r_2^2) \tag{13}$$

Thus, the potential is constant, and there is therefore no force everywhere inside the shell.

$$- \frac{\partial V}{\partial r'} = 0 \tag{14}$$

(iii) Potential at a point within the mass, $r_1 > r' > r_2$

Divide the spherical shell into two concentric shells with a spherical surface of radius r' (Fig. 3).

The potential of the outer shell V_o is, from (13)

$$V_o = 2\pi G\rho \, (r_1^2 - r'^2) \ ,$$

and that of the inner shell V_1 is, from (11)

$$V_i = \frac{4}{3} \pi G\rho \, (r'^3 - r_2^3) \cdot \frac{1}{r'}$$

Potential is a scalar quantity, so it can be simply added or substracted.

$$V = v_o + v_i = 2\pi G\rho \, (r_1^2 - \frac{r'^2}{3}) - \frac{4}{3} \pi G\rho \, r_2^3 \cdot \frac{1}{r'} \qquad (15)$$

$$-\frac{\partial V}{\partial r'} = \frac{4}{3} \pi G\rho \, (r' - \frac{r_2^3}{r'^2})$$

$$= G \frac{m_{r'}}{r'^2} - G \frac{m_{r2}}{r'^2}$$

$$= G \frac{m_i}{r'^2} \ , \qquad (16)$$

where $m_{r'}$, m_{r_2}, m_i denote the masses of the spheres of radius r' and r_2, and the mass of the inner shell respectively.

Thus, the attraction is only due to the inner shell and is the same as that of the whole inner mass concentrated at the centre.

2. Homogeneous Sphere

Let R be the radius of a sphere and ρ its density.

(i) For an outside point, r' > R.

Put $r_1 = R$ and $r_2 = 0$ in (11).

$$V = \frac{4}{3} \pi G\rho \frac{R3}{r'} \qquad (17)$$

$$= G \frac{\text{Total mass}}{r'}$$

$$-\frac{\partial V}{\partial r'} = G \frac{\text{Total mass}}{r'^2} \qquad (18)$$

(ii) For an inside point, $r' < R$.

Put $r_2 = 0$ in (15) and (16) (19)

$$V = 2\pi G\rho \left(R^2 - \frac{r'^2}{3}\right)$$

$$-\frac{\partial V}{\partial r'} = \frac{4}{3}\pi G\rho r' \quad (20)$$

$$= G \frac{\text{Mass of inner sphere}}{r'^2}$$

which is important to calculate the gravity in the earth.

3. Circular Disc

We consider the potential of a circular disc at a point on a central axis perpendicular to the disc (Fig. 4). Let a be the radius of the disc, dz its infinitely small thickness, and ρ its density.

Then an elementary mass dm at a point is given by $\rho r d\psi \cdot dr \cdot d_z$,

So the potential v is

$$V = G \int_{\psi=o}^{2\pi} \int_{r=o}^{a} \frac{\rho r d\psi \, dr \, dz}{\sqrt{r^2 + z'^2}}$$

$$= 2\pi G\rho \, dz \int_{o}^{a} \frac{rdr}{\sqrt{r^2 + z'^2}}$$

$$= 2\pi G\rho \, dz \left| \sqrt{r^2 + z'^2} \right|_{o}^{a}$$

$$= 2\pi G\rho \, dz \left(\sqrt{a^2 + z'^2} - z'\right) \quad (21)$$

$$\because \int \frac{X\,dX}{\sqrt{X^2 + A^2}} = \sqrt{X^2 + A^2}$$

The attraction at P' is

$$-\frac{\partial V}{\partial z'} = 2\pi G\rho \, dz \left(1 - \frac{z'}{\sqrt{a^2 + z'^2}}\right) \quad (22)$$

4. Right Circular Cylinder

Attraction of a right circular cylinder upon a point on its axis is directly obtained from the attraction of a disc by integration the expression with respect to z' (Fig. 5). From (22)

$$- \frac{\partial V}{\partial z'} = \int_o^h 2\pi G\rho \, dz \, (1 - \frac{H - z}{\sqrt{a^2 + (H - z)^2}})$$

$$= 2\pi G\rho \left| z + \sqrt{(H - z)^2 + a^2} \right|_o^h \quad \text{For integration, cf. (21)}$$

$$= 2\pi G\rho \, (h + \sqrt{(H - h)^2 + a^2} - \sqrt{H^2 + a^2})$$

$$= 2\pi G\rho \, (h + \ell_1 - \ell_2) \qquad (23)$$

When H = h, i.e., when the force at the centre of the upper surface of the cylinder is required,

$$- \frac{\partial V}{\partial z'} = 2\pi G\rho \, (H - \sqrt{H^2 + a^2} + a) \qquad (24)$$

5. Ring Cylinder

From (23) (Fig. 6), (outer cylinder) - (inner cylinder):

$$- \frac{\partial V}{\partial z'} = 2\pi G\rho \{ \sqrt{(H - h)^2 + a_2^2} - \sqrt{H^2 + a_2^2}$$

$$- \sqrt{(H - h)^2 + a_1^2} + \sqrt{H^2 + a_1^2} \} \qquad (25)$$

When H = h,

$$- \frac{\partial V}{\partial z'} = 2\pi G\rho \, (\sqrt{H^2 + a_1^2} - \sqrt{H^2 + a_2^2} + a_2 - a_1) \qquad (26)$$

The formulae (26) is used for the topographic corrections to be applied to observed values of gravity.

6. Infinitely Extended Plate

When the thickness h of a cylinder and the distance H of the attracted point are much smaller than its radius a, from (23)

$$-\frac{\partial V}{\partial z'} = 2\pi G\rho\{h + a\sqrt{1+\frac{(H-h)^2}{a^2}} - a\sqrt{1+\frac{H^2}{a^2}}\}$$

$$= 2\pi G\rho\ [h + a\{1 + \frac{1}{2}\ \frac{(H-h)^2}{a^2}\ \ldots\} - a\{1 + \frac{1}{2}\cdot\frac{H^2}{a^2}\ \ldots\}]$$

$$= 2\pi G\rho\ (h - \frac{2hH - h^2}{2a})$$

$$= 2\pi G\rho\ h\ (1 - \frac{2H-h}{2a}) \qquad (27)$$

When $a \to \infty$

$$-\frac{\partial V}{\partial z'} = 2\pi G\rho h \qquad (28)$$

The attraction of an infinite plate is dependent only on its thickness h and independent on the distance H of the attracted point from the plate. The above result is used for the Bouguer correction to the observed gravity.

7. Logarithmic Potential

Here is considered the potential of a line segment AB having a linear density λ. Taking the coordinate axes as Fig. 8, we have

$$\text{Potential of dm upon p}' = G\frac{\lambda dy}{e}$$

$$\text{Potential of AB upon p}' = V = G\int_{-a}^{b}\frac{\lambda dy}{\sqrt{(x-x')^2+y^2}}\ .$$

Since $\int\frac{dX}{\sqrt{X^2+A^2}} = \log\ (X + \sqrt{X^2+A^2})$,

$$V = G\lambda\left|\log\ (y + \sqrt{y^2+(x-x')^2}\right|_{-a}^{b}$$

$$= G\lambda\{\log\ (b + \sqrt{b^2+(x-x')^2}) - \log\ (-a + \sqrt{a^2+(x-x')^2})\}. \qquad (29)$$

When a, b (x - x'),

$$V = G\lambda[\log\{b + b(1 + \frac{(x-x')^2}{2b^2} - \frac{(x-x')^4}{8b^4} + \ldots)\}$$

$$-\log\{-a + a(1 + \frac{(x-x')^2}{2a^2} - \frac{(x-x')^4}{8a^4} + \ldots)\}]$$

$$= G\lambda[\log 2b\{1 + \frac{(x-x')^2}{4b^2} - \frac{(x-x')^4}{16b^4} + \ldots\ldots\}$$

$$-\log \frac{(x-x')^2}{2a}\{1 - \frac{(x-x')^2}{4a^2} + \ldots\ldots\}]$$

$$= G\lambda[\log 2b + \log\{1 + \frac{(x-x')^2}{4b^2} - \ldots\ldots\}$$

$$-\log\frac{(x-x')^2}{2a} - \log\{1 - \frac{(x-x')^2}{4a^2} + \ldots\ldots\}] .$$

Since $\log(1+X) = X - 1/2X^2 + 1/3X^3 - \ldots\ldots\ldots,$

$$V = G\lambda\{\log 2b + \frac{(x-x')^2}{4b^2} - \ldots\ldots\ldots$$

$$-\log\frac{(x-x')^2}{2a} + \frac{(x-x')^2}{4a^2} - \ldots\ldots\}$$

$$= 2G\lambda \log\frac{2\sqrt{ab}}{x-x'} + G\lambda\frac{(x-x')^2}{4}(\frac{1}{b^2} + \frac{1}{a^2}) \tag{30}$$

As the above expression is written as

$$V = 2G\lambda\log 2\sqrt{ab} - 2G\lambda\log(x-x') + G\lambda\frac{(x-x')^2}{4}(\frac{1}{b^2} + \frac{1}{a^2}) ,$$

$$+\frac{\partial V}{\partial x'} = +G\lambda\{\frac{2}{x-x'} - \frac{x-x'}{2}(\frac{1}{b^2} + \frac{1}{a^2})\} \tag{31}$$

This is the attraction of a line segment AB in the x' - direction when a and b are large compared with (x - x').

When a and b are infinity,

$$\frac{\partial V}{\partial x'} = G\lambda\frac{2}{x-x'}$$

If r denotes the perpendicular distance from p' to the line, i.e.,

$r = x - x'$,

$$- \frac{\partial V}{\partial r} = G\lambda \frac{2}{r}$$

and $V = -2G\lambda \log r$, (32)

This expression is called logarithmic potential, and is applied for the calculation of the attraction due to a body having a two dimensional form. The two dimensional form is a form having an infinite length in one direction and having everywhere the same cross section taken perpendiculary to that direction.

Suppose a two dimensional mass extending in finitely to the y' direction. Then its potential on a point p' (x', z') is given (Fig. 9)

$$V = -2G \rho \iint \log r \, dxdz \qquad (33)$$

where $r^2 = (x - x')^2 + (z - z')^2$ and dm corresponding to λ is $\rho \, dxdz$. So the gravitational force in the z' - direction due to the body is

$$\Delta g = \frac{\partial V}{\partial z'} = 2G\rho \frac{z - z'}{r^2} dxdz \qquad (34)$$

8. Attractions of two dimensional bodies

(a) Rectangular prism

Take the origin of the coordinate system at p' (Fig. 10), then $x' = 0$ and $z' = 0$.

$$\Delta g = 2G\rho \iint \frac{z}{r^2} dxdz$$

$$= 2G\rho \int_{x_1}^{x_2} \int_{z_1}^{z_2} \frac{z}{x^2 + z^2} dxdz ,$$

where x_1, x_2, z_1, z_2 are the coordinates of the four edges as shown in Fig. 10.

Since $\int \frac{Z}{X^2+Z^2}\,dZ = \frac{1}{2}\log(X^2+Z^2)$,

$$\Delta g = 2G\rho \int_{x_1}^{x_2} \left| \log\sqrt{(x^2+z^2)} \right|_{z_1}^{z_2} dx$$

$$= 2G\rho \int_{x_1}^{x_2} \log \frac{\sqrt{x^2+z_2{}^2}}{\sqrt{x^2+z_1{}^2}}\, dx$$

$$= 2G\rho \frac{1}{2} \left| \begin{matrix} x\log(x^2+z_2^2) - 2x + 2z_2 \tan^{-1}\frac{x}{z_2} \\ -x\log(x^2+z_1^2) + 2x - 2z_1 \tan^{-1}\frac{x}{z_1} \end{matrix} \right|_{x_1}^{x_2}$$

$$\because \int \log\sqrt{X^2+Z^2}\,dX = \frac{1}{2}\int \log(X^2+Z^2)\,dx$$

$$= \frac{1}{2}\{X\log(X^2+Z^2) - 2X + 2Z\tan^{-1}\frac{X}{Z}\}$$

$$\Delta g = G\rho\{x_2\log(x_2^2+z_2^2) - 2x_2 + 2z_2\tan^{-1}\frac{x_2}{z_2}$$

$$-x_2\log(x_2^2+z_1^2) + 2x_2 - 2z_1\tan^{-1}\frac{x_2}{z_1}$$

$$-x_1\log(x_1^2+z_2^2) + 2x_1 - 2z_2\tan^{-1}\frac{x_1}{z_2}\}$$

$$+x_1\log(x_1^2+z_1^2) - 2x_1 + 2z_1\tan^{-1}\frac{x_1}{z_1} .$$

Put $x_1^2+z_1^2 = r_1^2$ $\qquad \tan^{-1}\frac{z_1}{x_1} = \psi_1$

$x_2^2+z_1^2 = r_2^2$ $\qquad \tan^{-1}\frac{z_1}{x_2} = \psi_2$

$x_2^2+z_2^2 = r_3^3$ $\qquad \tan^{-1}\frac{z_2}{x_2} = \psi_3$

$x_1^2+z_2^2 = r_4^2$ $\qquad \tan^{-1}\frac{z_2}{x_1} = \psi_1$

then $\tan^{-1}\frac{x_2}{z_2} - \tan^{-1}\frac{x_1}{z_2} = (90^\circ - \psi_3) - (90^\circ - \psi_4) = \psi_4 - \psi_3$

$$\tan^{-1}\frac{x_2}{z_1} - \tan^{-1}\frac{x_1}{z_1} = (90^\circ - \psi_2) - (90^\circ - \psi_1) = \psi_1 - \psi_2 .$$

$$\Delta g = 2G\rho \{ x_2 \log \frac{r_3}{r_2} - x_1 \log \frac{r_4}{r_1} + z_2 (\psi_4 - \psi_3) - z_1 (\psi_1 - \psi_2) \} \quad (35)$$

(b) Vertical dike

If the lower boundary of a rectangular prism is extended downward to a considerable depth, that is, $z_2 \gg x_1, x_2$ (Fig. 11), we have approximately

$$z_2 (\frac{\pi}{2} - \psi_3) = x_2$$

$$z_2 (\frac{\pi}{2} - \psi_4) = x_1 ,$$

$$z_2 (\psi_4 - \psi_3) = x_2 - x_1$$

or $$\Delta g = 2 G \rho \{ x_2 \log \frac{r_3}{r_2} - x_1 \log \frac{r_3}{r_1} + (x_2 - x_1) - z_1(\psi_1 - \psi_2)\}$$

$$= 2G\rho \{ (x_2 - x_1) \log r_3 - x_2 \log r_2 + x_1 \log r_1 + (x_2 - x_1) - z_1 (\psi_1 - \psi_2) \} \quad (36)$$

As z_2 or r_3 increases, Δg increases. For $z_2 \to \infty$, Δg has an infinitely large value. However, since the increase depends upon $\log r_3$, Δg increase gradually with r_3.

(c) Vertical Fault

When one side of the rectangle is extended horizontally to a long distance, the form takes of a vertical fault (Fig. 12).

Putting in (35), $r_2 = r_3$ and $\psi_2 = \psi_3 = 0$, we have

$$\Delta g = 2G\rho (-x_1 \log \frac{r_4}{r_1} + z_2\psi_4 - z_1\psi_1) \quad (37)$$

When the thickness of the slab is small compared with its depth, (37) is much simplified.

Let $z_2 - z_1 = t$, and $\psi_4 = \psi_1 = \psi_1$

$$\log \frac{r_4^2}{r_1^2} = \log \frac{x_1^2 + z_2^2}{x_1^2 + z_1^2}$$

$$= \log \left(1 + \frac{2tz_1}{x_1^2 + z_1^2} + \frac{t^2}{x_1^2 + z_1^2}\right.$$

$$= \frac{zt^2{}_1}{x_1^2 + z_1^2} - \ldots\ldots$$

$$\because \quad \log (1 + X) = X - \frac{1}{2} X^2 + \frac{1}{3} X^3 - \ldots\ldots$$

$$\log \frac{r_4^2}{r_1^2} = \frac{2t}{z_1} \frac{1}{1 + \frac{x_1^2}{z_1^2}} ,$$

which can be neglected, if $t \ll z_1$

Therefore (37) become

$$\Delta g = 2G\rho t \psi \tag{38}$$

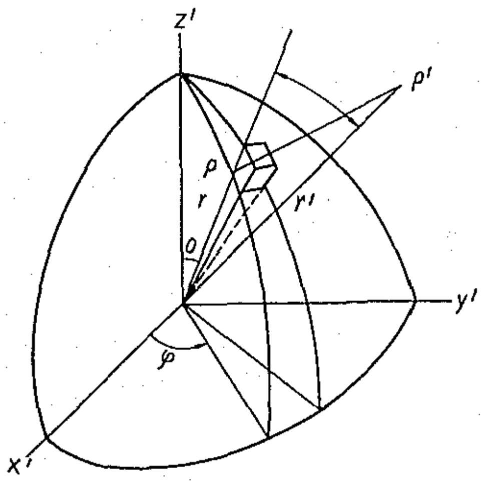


Fig. 1

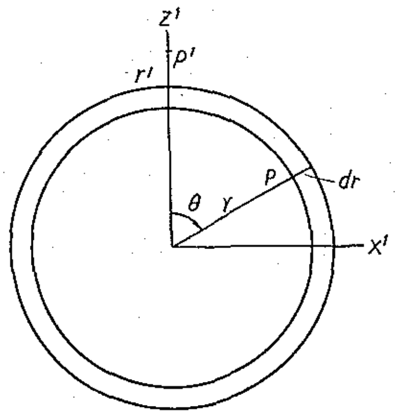


Fig. 2

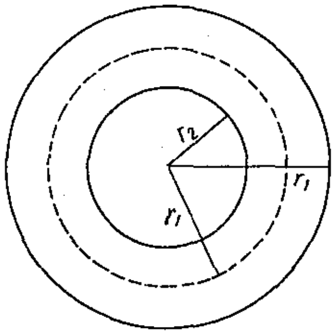


Fig. 3

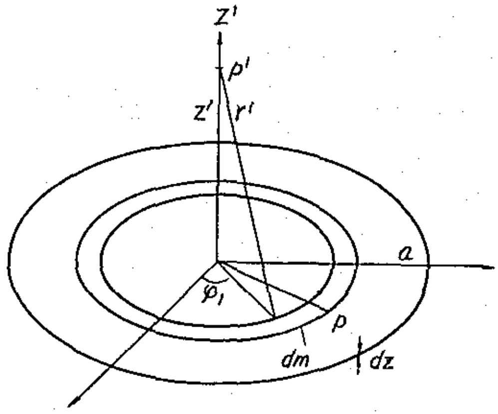


Fig. 4

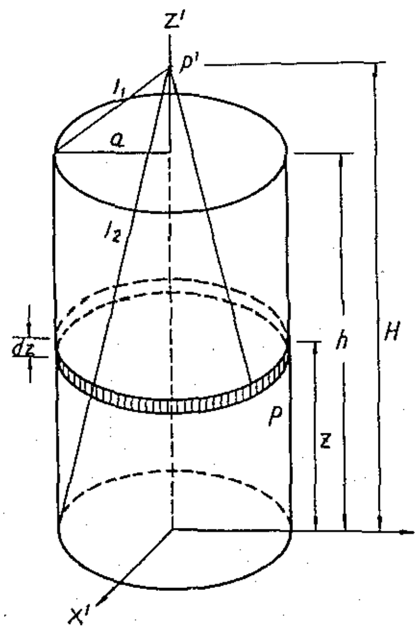


Fig. 5

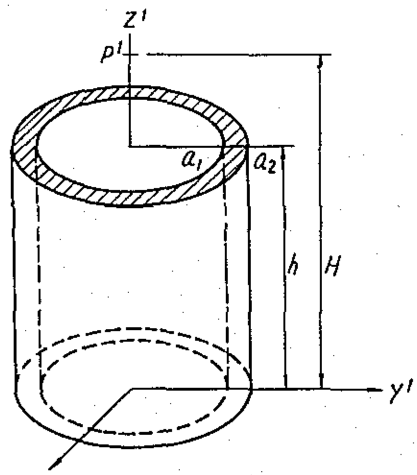


Fig. 6

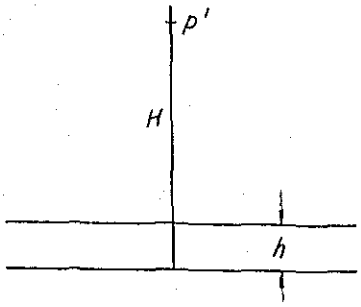


Fig. 7

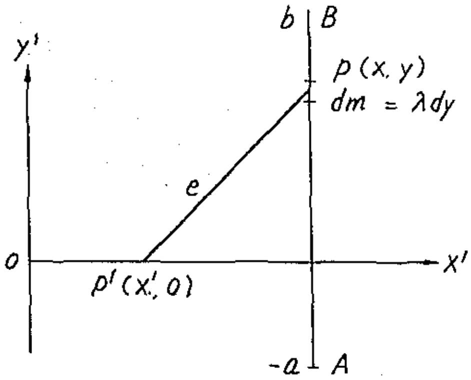


Fig. 8

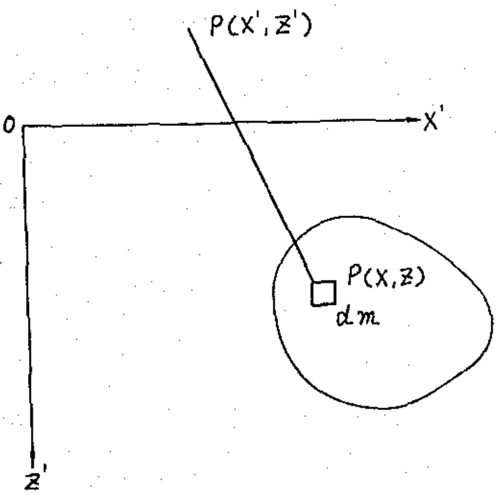


Fig. 9

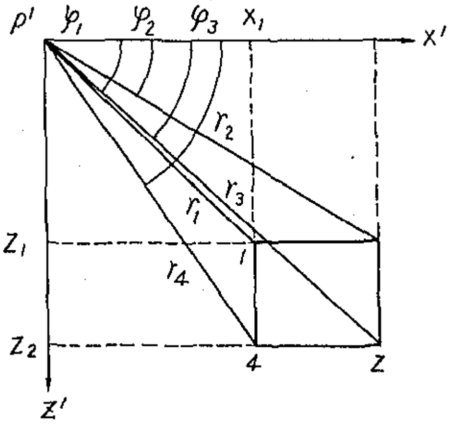


Fig. 10

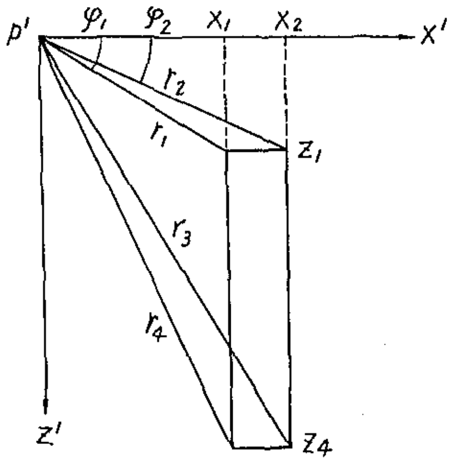


Fig. 11

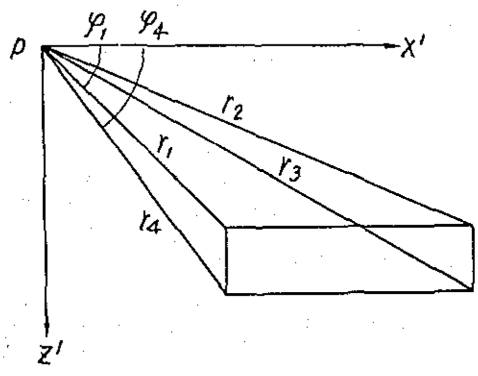


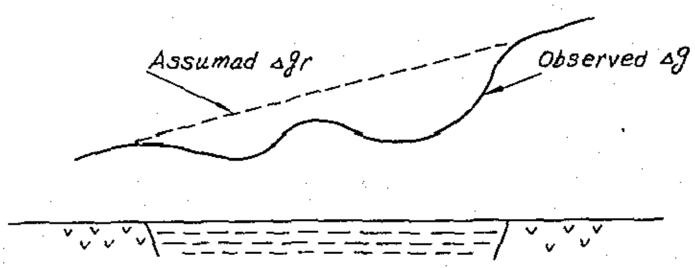
Fig. 12

V. Interpretation of gravity anomalies

1. Regional anomaly and local anomaly

Gravity anomalies measured by surveys are considered to consist of two anomalies: regional and local. The local anomaly often indicates anomalous mass distribution of special configuration or sometimes underground natural resources. Therefore the regional anomaly has to be removed from the observed anomaly. There are two methods for obtaining the regional field.

One is to determine the regional anomaly graphically from the gravity anomaly map, by taking, for instance, main profiles of the map. Fig. 1 shows such an example. In consistence with geology and other facts, the regional anomaly is considered adequately from the trend of the observed anomaly curve.



Another is the analytical method, in which for instance the following two methods are often used: determination of the mean value around a point where the local anomaly is required, and that of the distribution of regional field over a surveyed area.

(a) Method of mean value

To obtain the regional anomaly at a point P, a circle is drawn around the point P. The length of its radius is selected suitably, depending on the purpose of the survey. On the circle, say, sight points are taken, separated each other with an angle $2\pi/8$ subtended from the centre (Fig. 1).

The gravity anomalies at these eight points are determined from the isogal map of, say, Bouguer anomaly, thus Δg_1, Δg_2, ... Δg_8. We consider the mean value $\Delta g_r = \frac{1}{8} \sum_{n=o}^{8} \Delta g_n$ as the regional anomaly at P. Then the local anomaly at P is given $\Delta g_\ell = \Delta g - \Delta g_r$, where g is the observed anomaly at P read from the isogal map.

We calculate the local anomalies at many points, and draw the isogal map of local anomalies, which is useful for finding local underground structures.

When the mean value taken over a circular area around P is required, the area is divided into concentric circular zones and the mean value of each circular zone is calculated.

Practically, the area is divided with radial lines as well as concentric circles (Fig. 2).

In the example, radii of the concentric circles are r, 2r, 3r, 4r, and radial lines are taken every $2\pi/8$. There are 32 compartments of sectoral form. In every compartment the mean gravity anomaly is determined from the iso-anomaly map. In a circular zone we have eight values of anomaly which correspond respectively to the eight compartments. The mean value of the sight is considered to be the mean regional field of the circular zone at P.

Denote the mean value at P of the circular zone $(3r - 4r)$ by Δg_{3-4}, that of $(2r - 3r)$ by $\Delta \bar{g}_{2-3}$ and so on. The total areal mean of the observed anomalies at P is given as

$$\Delta g_p = \frac{1}{\pi(4r)^2} [\Delta \bar{g}_{3-4} \{\pi(4r)^2 - \pi(3r)^2\} + \ldots\ldots + \Delta g_{o-1} \pi r^2]$$

$$= \frac{1}{16} (7\Delta \bar{g}_{3-4} + 5\Delta \bar{g}_{2-3} + 3\Delta \bar{g}_{1-2} + \Delta \bar{g}_{o-1})$$

In these methods, the length of the radius of a circle around a point should be taken properly so that the mean value approximates the regional field at that point. If we take the radius large, the mean value seems to approach the regional field, but instead, other effect will enter the area.

As a method of area mean, we often divide a circular area into equal sectors, and estimate the mean anomaly Δg_m of each sector from the original isogal map. For instance, if sectors are eight as in Fig. 1, the regional anomaly at the centre is considered to be the arithmetic mean of the eight Δg_m.

If we divide the area around a point P into square compartments having sides parallel to NS and EW, the mean observed anomaly Δg_m of each compartment is estimated from the isogal map, and take their mean value for all compartments. In the case of Fig. 3, the mean value at P is

$$\Delta g_r = \frac{1}{16} \sum_{n=1}^{16} \Delta g_{mn}$$

when we have anomalies along a traverse, the regional effect can be in many cases represented by a straight line:

$$\Delta g_r = a_o \, i \, b_o x$$

where x is the distance of an observation point from a certain definite point, and a_o and b_o are constants to be determined by the method of least squares (Fig. 4).

(b) Method of finding distribution of regional field

The area concerned is divided into many square compartment (Fig. 5). The values of observed anomalies at the corners of the compartments are taken from the isogal map based on the observed anomalies.

For example, in Fig. 5 values of Δg at 25 cross points are read from the map. The coordinates (x, y) of the points are expressed by

(-2,-2)	(-1,-2)	(0,-2),	(1,-2),	(2,-2),
(-2,-1)	(-1,-1)	(0,-1),	(1,-1),	(2,-1),
- -	- -	- -	- -	- -
(-2,2),	(-1,2),	(0,2),	(1,2),	(2,2).

We assume that the regional field is represented by

$$\Delta g = a + px + qy$$

Giving 25 values of Δg and the coordinates x, y of the points where Δg are taken, we have 25 observation equations, from which we can determine a, p and q. Then the regional field is given by the f formula

$$\Delta gr = a_o + p_o x + q_o y ,$$

where suffix o means the value determined by the method of least squares.

$$\Delta g_r \text{ ot } (x_1, y_k) - (a_o + p_o x + q_o y_k)$$

Without using the method of least squares, a_o, p_o, and q_o can be determined. Take the mean value of Δg at (1,1) and (1,5) and consider it as the regional value at (1,3). Similarly, the values at (11,3), (111,3), ... (V,3) are determined as the regional values at these points. They are plotted on a graph paper as in Fig. 5.

The distribution of the dotts is assumed to be linear, and the slope of the line and its intercept at Δg_r axis are determined as α and a_o. In the above equation, when y = o,

$$\Delta g_r = a_o + p_o x ,$$

which corresponds to the line.

Actually

$$p_o = \frac{\partial \Delta g_r}{\partial x} = \tan \alpha$$

So from the line in the graph paper, we can determine p_o as tan α.

In the same way, take the mean value of Δg at (1,1) and (V,1), and consider it to be the regional field at (III,1). Similarly the regional fields at (III,2), (III,3),........ (III,5) are determined. Plotting these values in the graph paper just like the above, we can determine a_o and q_o, where q_o = tan ρ. In this case a_o must be different from the value previously obtained. So the mean of the two a_o is taken as the constant of the necessary formula including both the terms of p_o and q_o.

When the regional field is not represented by a linear equation, we assume a quadratic equation:

$$\Delta g_r = a_{oo} + a_{1o} + a_{1o}x^2$$
$$+ a_{o1}y + a_{o2}y^2 + a_{11}xy$$

The coefficients a's are determined by the method of least squares.

When discrepancy of observed anomalies is conspicuous, a method of smoothing is often used. A simple method is given as follows.

The observed anomalies at three consecutive points on a traverse are taken for the smoothing (Fig. 6), the spacing of the observation points being supposed to be approximately equal.

Suppose

$$2S\ \Delta g_m = (S - \delta S)\ \frac{1}{2}\ (\Delta g_1 + \Delta g_2) + (S + \delta S)\ \frac{1}{2}\ (\Delta g_2 + \Delta g_3)\ ,$$

where Δg_m is the smoothed value at the middle point between P_1 and P_3.

$$\Delta g_m = \frac{1}{4}\ (\Delta g_1 + 2\Delta g_2 + \Delta g_3) - \frac{1}{4}\ \frac{\delta S}{S}\ (\Delta g_1 - \Delta g_3)$$

If P_2 is taken just at the centre between P_1 and P_3, i.e., $\delta S = 0$ the smoothed Δg_m is given by $1/4\ (\Delta g_1 + 2\Delta g_2 + \Delta g_3)$. When $\delta S \ll S$ and $\Delta g_1 - \Delta g_3$ is not large, the correctional term can be neglected.

2. Interpretation for simple bodies

From isogal map showing local anomalies we try to find out underground structures, referring to geology, drilling results etc.

In general, gravity highs often indicate anticlines, igneous intrusions, metallic ores, etc. Gravity lows often show sedimentary deposits which are lighter than the surroundings. Strips of steep increase of gravity are often attributed to faults.

Next we deal with the interpretation of gravity anomalies caused by bodies of simple forms.

(a) Sphere

From (18) of the preceeding chapter, the attraction due to a spher at P' is given by

$$G\,4/3\pi R^3\,\delta/r'^2 ,$$

where R is the radius of the sphere, δ its density, $r'^2 = x'^2 + z^2$ as shown in Fig. 7.

Denoting its z-component by Δg_z we have

$$\Delta g_z = G\,\frac{4}{3}\,\pi R^3\,\delta\,\frac{1}{x'^2+z^2}\cdot\frac{z}{r'}$$

$$= G\,\frac{4}{3}\,\pi\,\frac{R^3}{z^1}\,\delta\,\frac{1}{\left(1+\frac{x'^2}{z^2}\right)^{3/2}} \qquad (1)$$

when $x' = 0$, Δg_z has a maximum value Δg_{max}.

$$\Delta g_{max} = G4/3\pi\,\frac{R^3}{z^2}\,\delta \qquad (2)$$

So generally

$$\Delta g_z = \Delta g_{max}\,\frac{1}{\left(1+\frac{x'^2}{z^2}\right)^{3/2}}$$

When $\Delta g_z = 1/2 \ \Delta g_{max}$,

$$\frac{1}{2} = \frac{1}{(1 + \frac{x'^2}{z^2})^{3/2}}$$

$$\therefore Z = 1.305 \ x'_{1/2} \qquad (3)$$

where $x'_{1/2}$ is the distance between the two points having Δg_{max} and $1/2\Delta g_{max}$ respectively.

Knowing $x'_{1/2}$ we can estimate the depth z of the centre of the sphere. If δ is known R can be found from (2), being here the difference between the density of the sphere and of the surroundings.

(b) Vertical rod

Consider a vertical thin rod having a density λ whose upper and lower ends are at z_1 and z_2. Its gravity effect is given by

$$\Delta g = G \int_{z_1}^{z_2} \frac{\lambda \, dz}{r'^2} \cdot \frac{z}{r'}$$

$$= G\lambda \int_{z_1}^{z_2} \frac{z}{(x'^2 + z^2)^{3/2}} \, dz$$

Since

$$\int \frac{x \, dx}{(x^2 + A^2)^{3/2}} = - \frac{1}{\sqrt{x^2 + A^2}}$$

$$\Delta g = G\lambda \left| \frac{1}{\sqrt{x'^2 + z^2}} \right|_{z_1}^{z_2}$$

$$= G\lambda \left(\frac{1}{\sqrt{x'^2 + z_1^2}} - \frac{1}{\sqrt{x'^2 + z_2^2}} \right) \qquad (4)$$

When z_2 is ∞

$$\Delta g = G\lambda \frac{1}{\sqrt{x'^2 + z_1^2}} \qquad (5)$$

In (5), when x = 0

$$\Delta g_{max} = G\lambda \frac{1}{2} \qquad (6)$$

Denoting that $x'_{1/2}$ is the distance of a point where g take $1/2\Delta g_{max}$, we have

$$\frac{1}{2}\Delta g_{max} = G\lambda \frac{1}{\sqrt{x'^2_{1/2} + z_1^2}} = \frac{G\lambda}{Z_1} \frac{1}{\sqrt{\left(\frac{x'_{1/2}}{z_1}\right)^2 + 1}}$$

$$\left(\frac{x'_{1/2}}{z_1}\right)^2 + 1 = 4, \qquad X'_{1/2} = \sqrt{3}\ Z_1 \qquad (7)$$

Therefore if we find the distance $x'_{1/2}$ from the two points Δg_{max} and $1/2\,\Delta g_{max}$, the depth z, of the semi-infinite vertical rod can be found (Fig. 8).

The density λ is obtained from (6) as

$$\lambda = \frac{x'_{1/2}}{\sqrt{3}} \frac{\Delta g_{max}}{G}$$

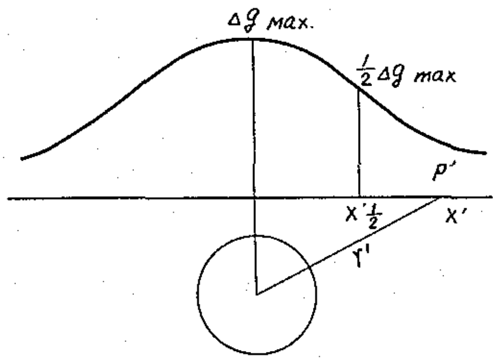


Fig. 7

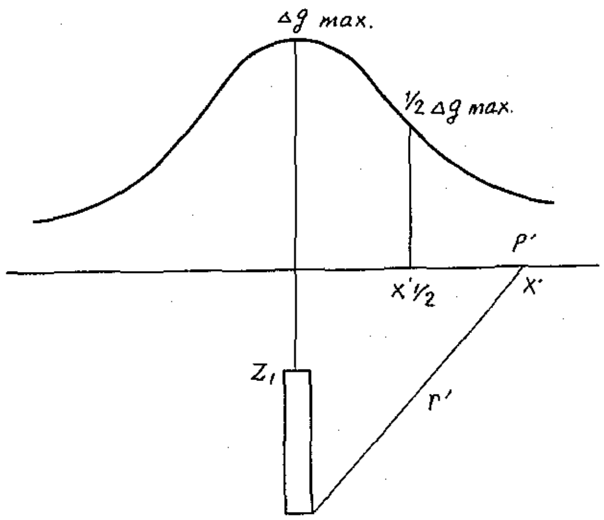


Fig. 8

(c) Horizontal infinite cylinder

First we calculate the attraction Δg of a horizontal infinite cylinder on a point just above its horizontal axis.

From (34) of the preceding chapter,

$$\Delta g = \frac{\partial V}{\partial Z'} = 2\sigma \int \frac{Z - Z'}{e^2} \, dm$$

where $e^2 = r'^2 + r^2 - 2r' r \cos(\pi - \theta)$ Cf. Fig. 9

$dm = \sigma r \, d\theta \, dr$,

$z = r \cos \theta$,

$-z' = r.'$ (r' is taken positive)

$$\Delta g = 4G\sigma \int_0^k r dr \int_0^\pi \frac{r' + r \cos\theta}{r'^2 + r^2 + 2r' r \cos\theta} \, d\theta \ ,$$

where R is the radius of the cross-section of the cylinder.

The integral with respect to θ, I_θ, is transformed as follows:

$$I_\theta = \int_0^{\pi - \Delta\theta} \frac{r' + r \cos}{r'^2 + r^2 + 2r' r \cos\theta} \, d\theta + \frac{r' + r \cos\pi}{r'^2 + r^2 + 2r' r \cos\pi} \Delta\theta$$

where θ is taken to be so small that the integrand may be regarded constant for the interval from $\pi - \Delta\theta$ to π

For the first term we use the following formula:

$$\int \frac{a' + b' \cos' \theta}{a + b \cos \theta} = \frac{b' \theta}{b} + \frac{a'b - ab'}{b} \int \frac{d\,\theta}{a + b \cos \theta} \quad ,$$

where

$$\int \frac{d\,\theta}{a + b \cos \theta} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{\sqrt{a^2 - b^2} \tan \frac{\theta}{2}}{a + b} \quad ,$$

$$-\pi < G < \pi$$

First term of

$$I_\theta = \frac{1}{2r} (\pi - \Delta\theta) + \frac{1}{r'} \tan^{-1} \{ \frac{r' - r}{r' + r} \tan (\frac{\pi}{2} - \frac{\Delta\theta}{2}) \} \; ,$$

where $\tan \frac{\pi}{2} - \frac{\Delta\theta}{2} = \infty$

$$\text{First term} = \frac{\pi}{r'} - \frac{\Delta\theta}{2r'}$$

While

$$\text{Second term} = \frac{1}{r' - r} \Delta\theta$$

Let $\Delta\theta$ approach 0,

$$I_\theta = \frac{\pi}{r'}$$

$$\Delta g = \frac{4 \pi G \sigma}{r'} \int_o^R r\,dr$$

$$= 2 \pi R^2 G \sigma \frac{1}{r'} \quad (8)$$

When p' is not on the z' -axis, the vertical component Δg_z of the above Δg has to be taken.

$$\Delta g_z = 2 \pi R^2 G\sigma \frac{1}{r'} \frac{2}{r'}$$

$$= 2 \pi R^2 G\sigma \frac{2}{x'^2 + z^2}$$

$$\text{or} = \quad = 2 \pi R^2 G\sigma \frac{1}{2} (\frac{1}{1 + \frac{x'^2}{x^2}}) \quad (9)$$

When $x' = 0$, Δg_z has a maximum value:

$$\Delta g_{max} = 2\pi R^2 G\sigma \frac{1}{2} \quad (10)$$

$$\Delta g_z = \Delta g_{max} \left(\frac{1}{1 + \frac{x'^2}{z^2}} \right)$$

At the point $x'_{1/2}$ where $\Delta g_z = 1/2 \, \Delta g_{max}$

$$1 + \frac{x'^2_{1/2}}{z^2} = 2$$

$$x'_{1/2} = 2$$

The depth z of the centre of the cylinder can be found as $x'_{1/2}$, which is determined from the distance between the two points having Δg_{max} and $1/2 \, \Delta g_{max}$ respectively.

If δ is known, R is given from (10).

3. Two dimensional body having an irregular cross-section

(a) A method of using graticule

The effect of a two dimensional body is given by

$$\Delta g = 2 G\sigma \iint \frac{(2 - 2')}{r^2} \, dx \, dz$$

where $r^2 = (x - x')^2 + (z - z')^2$

If we take the attracted point P' at the origin of coordinates, $x' = 0$ and $z' = 0$,

$$\Delta g = 2G\sigma \iint \frac{2}{r^2} \, dx \, dz$$

where $r^2 = x^2 + z^2$

When the cylinderical coordinates are used (Fig. 12).

$$x = r \cos \psi$$

$$z = r \sin \psi$$

$dxdz = dr.rd\psi$

$$\Delta g = 2G\sigma \int_r \int_\psi \sin\psi \, dr d\psi \qquad (16)$$

The integration is generally difficult, because the limits of integration of r relate to the value of ψ and vice versa. Therefore, to simplify the problem, we take a small portion of the body which has the same limits of r even if the value differs. Thus, for ψ, the lower and upper limits of r are r and $r + \Delta r$; for $\psi + \Delta\psi$, the limits are also r and $r + \Delta r$ (Fig. 13). Then the attraction of the portion on the point p' is given by

$$\delta\Delta g = 2G\sigma \int_r^{r+\Delta r} \Delta r \int_\psi^{\psi+\Delta\psi} \sin\psi \, d\psi$$

$$= -2G\sigma \left| r \right|_r^{r+\Delta r} \left| \cos\psi \right|_\psi^{\psi+\Delta\psi}$$

$$= -2G\sigma \{ (r+\Delta r) - r \} \{ \cos(\psi + \Delta\psi) - \cos\psi \}$$

where ψ and $\psi+\Delta\psi$ are the limits of ψ, and r and $r + \Delta r$ are the limits of r.

Therefore, for the whole mass

$$\Delta g = -2G\sigma \sum_r \sum_\psi \{(r+\Delta r) - r\} \{\cos(\psi+\Delta\psi) - \cos\psi\}$$

If the mass is divided into many portions, say n, so that each portion has a constant gravity effect $\delta\Delta, g_c$ on P'. The effect of all the mass on P' is obtained as

$$\Delta g = \delta\Delta g_c$$

If a quadrant is divided into many small compartments, each of which has a dimension of $\Delta r = 1$ cm, $\cos\psi - \cos(\psi + \Delta\psi) = 0.05$, each compartment having a density of 1 exerts the point P' with an acceleration 6.67×10^{-6} mgal (cf.(17). Fig. 14 shows such compartments on a reduced scale.

Practically, the figure with $\Delta r = 1$ cm is drawn on a transparent sheet and put it on a section showing an assumed underground body, coinciding

0 - point of the sheet with the attracted point P' and the line ox' with the horizontal surface. By counting the number of compartments which occupy the cross-section of the body, we can find Δg at P' due to the two dimensional body.

When the section is drawn on a scale of 1 : S and the density contrast is σ, the effect of one compartment is $6.67 \times 10^{-6} \times S \times \sigma \times$ mgal.
For instance, if, the section is drawn on $1 : 10^4$, and if density contrast is 0.2,

$$\delta\Delta g_c = 6.67 \times 10^{-6} \times 10^4 \times 0.2$$

The graticule is drawn as follows. Around a centre 0, many concentric circles are drawn with radii 1 cm., 2 cm., 3 cm.,........, the number of circles being taken properly according to the object.

As to ψ, for x' -axis $\psi = 0$. $\cos\psi = 1$.
Since $\cos\psi - \cos(\psi + \Delta\psi) = 0.05$, $\psi + \Delta\psi = 18°12'$,
where $\psi = 0$. Therefore, the first radial line angles 18°12' with x' -axis.

For the first radial line, $\psi = 18°12'$. So $\cos\psi \approx 0.95$.
Since $\cos\psi - \cos(\psi , \Delta\psi) = 0.05$, $\cos(\psi + \Delta\psi) = 0.90$.

$$\psi + \Delta\psi = 25°51'$$

$$\text{or} \quad \Delta\psi \approx 7°39'.$$

Therefore, the second radial line makes an angle of 25°51' with the x' -axis, and it angles 7°39' with the first radial line.

(b) Estimation of depth

We suppose that a sediment of density is deposited on the underlying denser bed rock of density σ, and indicate the minimum depth of the surface of the bed rock by z, and its maximum depth by z_2 (Fig. 15).

When the infinitely extending slab of thickness $(z_2 - z_1)$ is wholly made of σ', its gravity anomaly on the ground surface shows a maximum value

$(\Delta g)_{max}$, while when the slab is wholly made of the anomaly has a minimum value $(\Delta g)_{min}$, i.e.,

$$(\Delta g)_{max} = (\Delta g)_{min} + 2\pi G \Delta\sigma (z_2 - z_1) \qquad (18)$$

where $\Delta\sigma = \sigma' - \sigma$

Any observed anomaly Δg has a value between the above two extreme values i.e.,

$$(\Delta g)_{max} > \Delta g > (\Delta g)_{min}$$

or

$$(\Delta g)_{max} - (\Delta g)_{min} > g - (\Delta g)_{min}$$

This inequality holds also when Δg takes a maximum value,

$$(\Delta g)_{max} - (\Delta g)_{min} > \Delta g_{max} - \Delta g_{min},$$

where the two notations of the right hand side indicate respectively the maximum and the minimum observed anomalies.

Referring to (18), we have

$$z_2 - z_1 > \frac{\Delta g_{max} - \Delta g_{min}}{2\pi G \Delta\sigma} \qquad (19)$$

So $z_2 - z_1$ is larger than the depth difference calculated from the maximum change of Δg in the field.

The depth z of the bed rock is often calculated from Δg and Δg_{max} by the following formula

$$z - z_1 = \frac{\Delta g_{max} - \Delta g}{2\pi G \Delta\sigma} \qquad (20)$$

here Δg being the observed anomaly at the point where z is to be sought.

When the maximum depth is concerned, the inequality (19) has to be taken into consideration. (19) and (20) can be of course applied not only to a two dimensional surface, but also to a general surface.

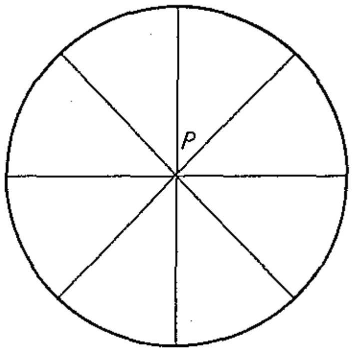


Fig. 1

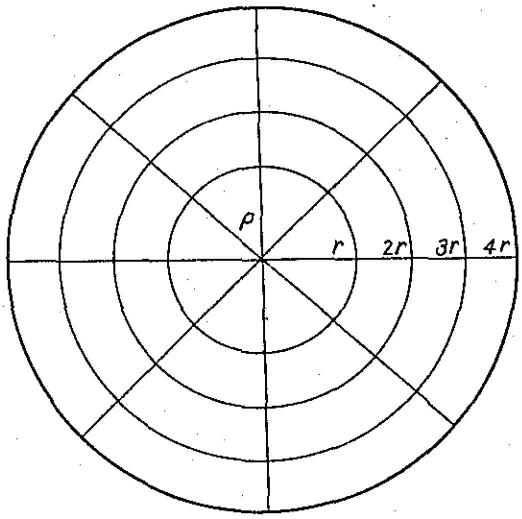


Fig. 2

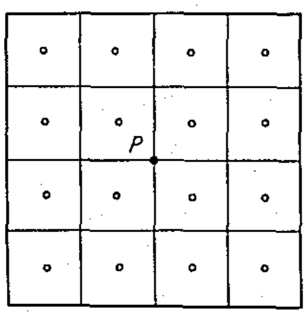


Fig. 3

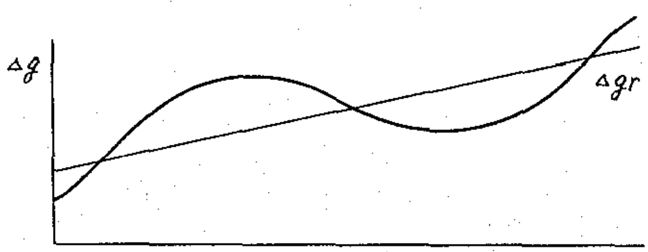


Fig. 4

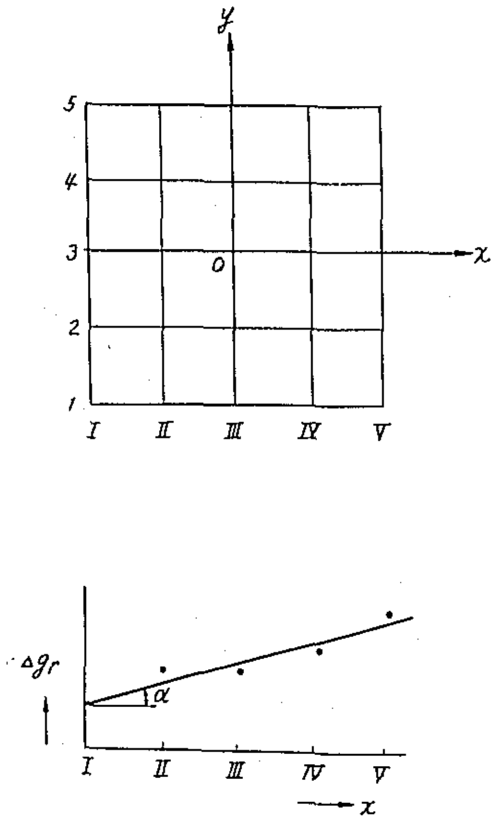


Fig. 5

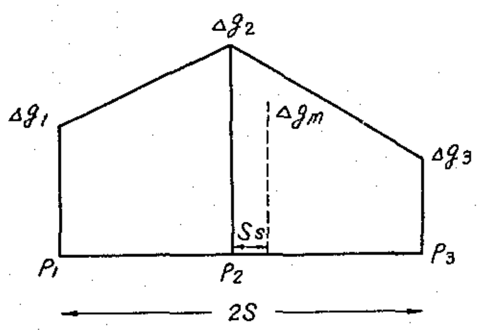


Fig. 6

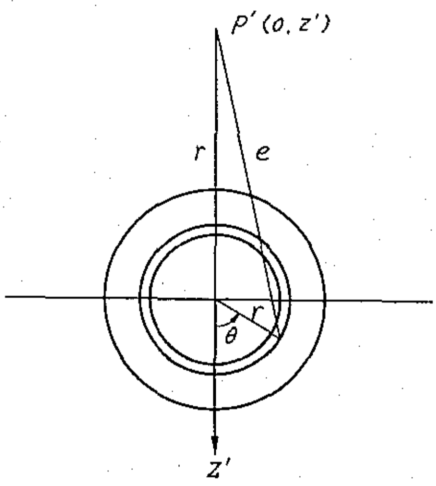


Fig. 9

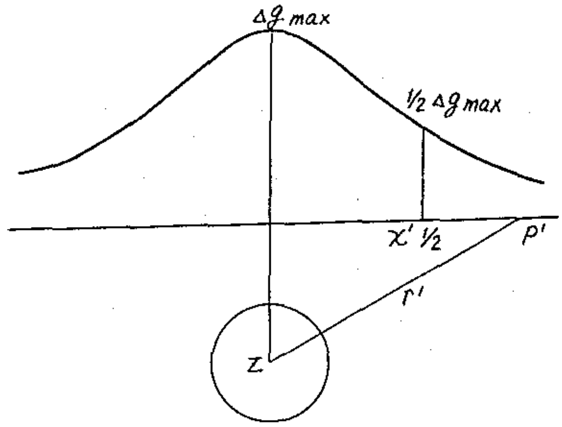


Fig. 10

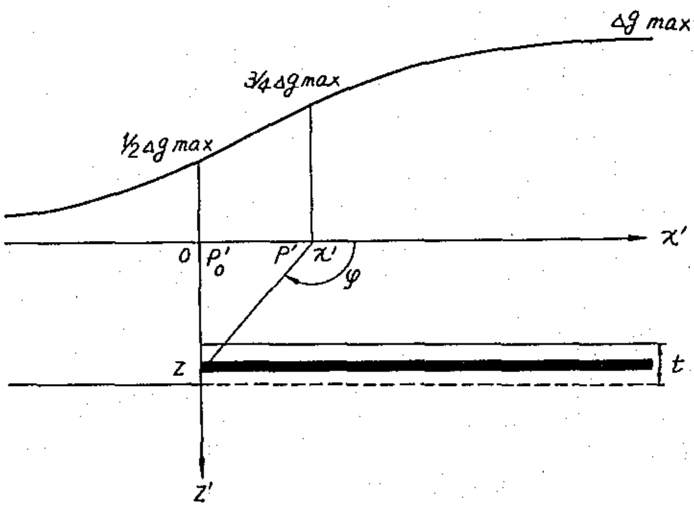


Fig. 11

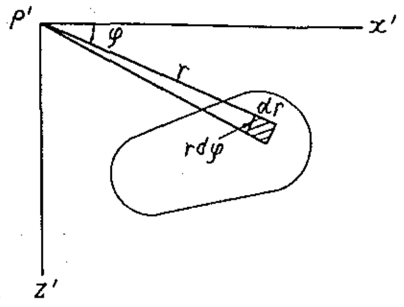


Fig. 12

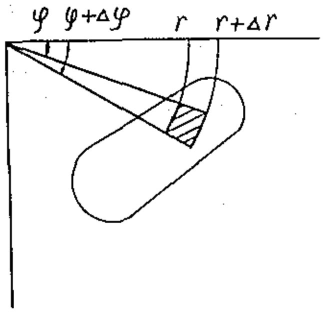


Fig. 13

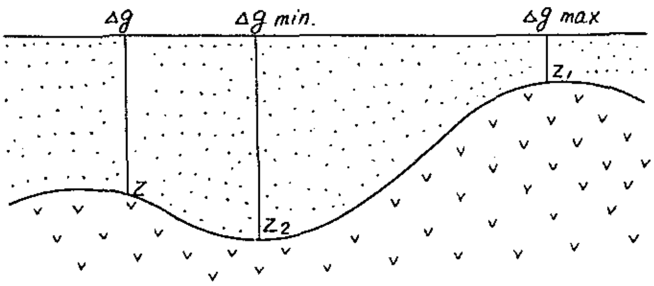


Fig. 15

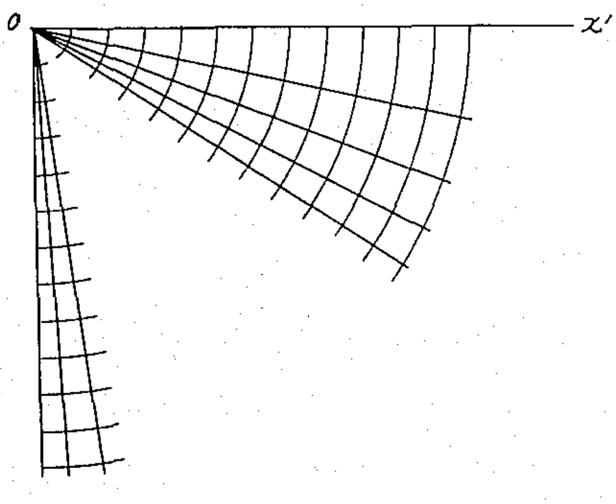


Fig. 14

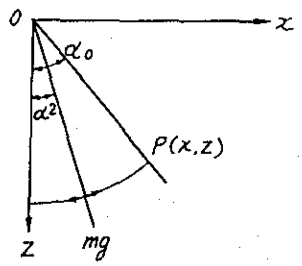
Theory of Pendulum

Appendix

1. Mathematical Pendulum:

The classic method to measure gravity is to observe the period of a pendulum. A mathematical pendulum is the one having a mass that is suspended freely by a fine massless thread under the influence of gravity and swings about a horizontal axis through the point of suspension.

Let O be the point of suspension of a pendulum of length 1, P the end point of the pendulum with a mass m, α the phase angle of the pendulum, then the tangential component toward + x of the oscillation force considered from the motion of the pendulum is given:



$$\frac{md^2S}{dt^2} = \frac{mld^2\delta}{dt^2}$$

Fig. 1

The above equality can be obtained as follows:

$$\frac{ds}{dt} = V = \frac{\ell . d\delta}{dt} = \ell\ w$$

$$\frac{d^2s}{dt^2} = \frac{\ell\, d^2\delta}{dt^2}$$

in which $\frac{ds}{dt}$ and $\frac{d}{dt}$ mean the velocity of the mass and its angular velocity considered toward the counterclockwise direction, and both the accelerations $\frac{d^2\delta}{dt^2}$ and $\frac{d^2\delta}{dt2}$ have negative values toward that direction.

The above tangential force must be equal to the negative of the tangential component of the gravitational force acting on m.

Therefore

$$\frac{m\ell d^2\delta}{d\ell 2} = mg \sin \alpha$$

$$\frac{d^2\delta}{dt^2} + \frac{g}{\ell} \sin \alpha = C \qquad (1)$$

As to the normal force, it can be disregarded. Because the outward forces, i.e. mg cos α and the centrifugal force $m\ell w^2$ are balanced by the tension of the suspending thread.

(1) is the differential equation of the motion of a mathematical pendulum.

To obtain g with l, t and δ, we integrate the above equation. First, it is multiplied with $\frac{2d\delta}{dt}$ dt

$$\frac{2d\delta}{dt} \frac{d^2\delta}{dt^2} dt = -2 \frac{g}{\ell} \sin\alpha \frac{d\delta}{dt} dt$$

By integration,

$$(\frac{d\delta}{dt})^2 = 2 \frac{g}{\ell} \cos\alpha + C$$

Let α_o be the maximum phase angle i.e. $\alpha = \alpha_o$ when $\frac{d\alpha}{dt} = 0$. Then the above integration constant C is given as

$$C = -2 \frac{g}{\ell} \cos \alpha$$

$$(\frac{d\delta}{dt})^2 = 2 \frac{g}{\ell} (\cos\alpha - \cos \alpha_o)$$

$$dt = \sqrt{\frac{1}{2g}} \cdot \frac{d\alpha}{\sqrt{\cos\alpha - \cos\alpha_o}}$$

Since generally $\cos\alpha = 1 - 2 \sin \frac{\alpha o}{2}$, we have

$$dt = \frac{1}{2} \sqrt{\ell/g} \frac{dx}{\sqrt{\cos\alpha - \cos \alpha o}}$$

To integrate this, we introduce a new variable ψ, thus

$$\sin\frac{\alpha}{2} = \sin\frac{\alpha o}{2} \sin\psi$$

Then, by differentiating this equality with respect to ψ

$$\frac{1}{2} \cos\frac{\alpha}{2} \frac{d\delta}{d\psi} = \sin\frac{\alpha o}{2} \cos\psi,$$

or

$$d\delta = \frac{2 \sin\frac{\alpha o}{2}}{\cos\frac{\alpha}{2}} \cos\psi \, d\psi$$

$$dt = \sqrt{\ell/g} \frac{d\psi}{\cos\frac{\alpha}{2}}$$

Since $\cos\frac{\delta}{2} = \sqrt{1 - \sin^2\frac{\alpha o}{2} \sin^2\psi}$ from the definition of ψ,

$$dt = \sqrt{\ell/g} \frac{d\psi}{\sqrt{1 - \sin^2\frac{\alpha o}{2} \sin^2\psi}}$$

Limits of integration are

$$\psi = o \quad \text{for} \quad \alpha = o$$

$$\psi = \frac{\pi}{2} \quad \text{for} \quad \alpha = \alpha_o$$

$$t = \sqrt{\ell/g} \int_{\ell}^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2\frac{\alpha o}{2} \sin^2\psi}} \tag{2}$$

where t is the half of the half period. The integral is an elliptic integral and is calculated as follows.

If αo is small, the integrand can be expanded by binomial series. Thus putting $\sin^2\frac{\alpha o}{2} = k^2$,

$$\frac{1}{\sqrt{1 - k^2 \sin^2\psi}} = 1 + \frac{1}{4} k^2 - \frac{1}{4} k^2 \cos 2\psi$$

$$+ \frac{9}{64} k^4 - \frac{3}{16} k^4 \cos 2\psi + \frac{3}{64} k^4 \cos 4\psi + \ldots..$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha_o}{2}\sin^2\psi}} = (1+\frac{1}{4}k^2+\frac{9}{64}k^4)\int_0^{\frac{\pi}{2}} d\psi$$

$$-(\frac{1}{4}k^2+\frac{3}{16}k^4)\int_0^{\frac{\pi}{2}} \cos 2\psi \ d\psi$$

$$+\frac{3}{64}k^4\int_0^{\frac{\pi}{2}} \cos 4\psi \ d\psi$$

But

$$\int_0^{\frac{\pi}{2}} d\psi = \frac{\pi}{2} ,$$

$$\int_0^{\frac{\pi}{2}} \cos 2\psi d\psi = \left|\frac{1}{2}\sin 2\psi\right|_0^{\frac{\pi}{2}} = 0 ,$$

$$\int_0^{\frac{\pi}{2}} \cos 4\psi d\psi = \left|\frac{1}{4}\sin 4\psi\right|_0^{\frac{\pi}{2}} = 0 ,$$

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha_o}{2}\sin^2\psi}} = (1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\ldots)\frac{\pi}{2}$$

So (2) is

$$t = \frac{\pi}{2}\sqrt{\frac{\ell}{g}}(1+\frac{1}{4}\sin^2\frac{\alpha_o}{2}+\frac{9}{64}\sin^4\frac{\alpha_o}{2}+\ldots\ldots) ,$$

t is the time that the pendulum moves from the vertical ($\alpha = 0$) to the highest position ($\alpha = \alpha_o$). If the whole period is indicated by T, T = 4t.

As the fourth order of $\sin\frac{\alpha_o}{2}$ is practically neglected,

$$T = 2\pi\sqrt{\frac{\ell}{g}}(1+\frac{1}{4}\sin^2\frac{\alpha_o}{2}) ,$$

where

$$\sin^2\frac{\alpha_o}{2} = \{\frac{\alpha_o}{2}-(\frac{\alpha_o}{2})^3\frac{1}{2.3}+(\frac{\alpha_o}{2})^5\frac{1}{2\cdot3\cdot4\cdot5}-\ldots\}^2$$

$$= \frac{\alpha_o^2}{4}-\frac{\alpha_o^4}{48}+\ldots\ldots \qquad 2\cdot3$$

$$T = 2\pi\sqrt{\frac{\ell}{g}}(1+\frac{\alpha_o^2}{16}) \tag{3}$$

As the oscillation period of a pendulum, we use the period for infinitely small amplitude ($d_0 = 0$). Indicating the period by T_0, we have

$$T_0 = 2\pi\sqrt{\ell/g}$$

$$T = T_0 \left(1 + \frac{\alpha_0^2}{16}\right) \quad (4)$$

$$T_0 = T - \frac{\alpha_0^2}{16} T \quad (15)$$

The required period $\bar{T}o$ is obtained from the observed period T (5) by adding a correction $-\frac{\alpha_0^2}{16}$ T, which is called the amplitude correction for the correction, for instance, as a simple method the average value of the two amplitudes at the start of the observation and at its end is taken.

From (4) we see that g is obtained from the period of oscillation if ℓ is measured accurately, thus

$$g = 4\pi^2 \frac{\ell}{T_0^2} \quad (6)$$

2. Physical Pendulum

We consider the vibration of a rigid body about a horizontal axis, i.e., of a physical pendulum. In Fig. 2, let 0 denote the horizontal axis, dm an elementary mass of the body at (x, z), r the distance between the axis and dm, (x_G, z_G), the centre of gravity G of the body, M its total mass, and h the distance of G from the axis 0. Here G is not generally in the xz plane.

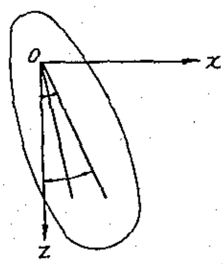


Fig. 2

From (1) we have the following equation for dm.

$$dm\ r\ \frac{dw}{dt} = -dmg\ \frac{x}{r} \tag{7}$$

$$r^2\ \frac{dw}{dt}\ dm = -g\ x\ dm$$

where $r^2 dm$ is the moment of inertia of the body about the axis 0.

The left hand side of (7) indicates the force acting on dm which is considered only from the actual motion of the pendulum. But the right hand side of the equation means the force which originates or maintains the actual motion.

By the definition of the centre of gravity,

$$\int x\ dm = x_G M$$

Therefore

$$\frac{dw}{dt} \int r^2 dm = -g x_G M$$

$$= gMh \sin \alpha, \tag{8}$$

where α denotes the phase angle of OG and h the distance OG.

Consider an axis through G parallel to the suspension axis and the distance of dm from the axis to be ρ , thus

$$\rho^2 = (x - x_G)^2 + (z - z_G)^2$$

Introducing

$$x = x_G + (x - x_G)$$

$$z = z_G + (z - z_E)^2$$

we have

$$r^2 = x^2 + z^2$$

$$= x_G{}^2 + z_G{}^2 + (x - x_G)^2 + (z - z_G)^2$$

$$+ 2x_G\ (x - x_G) + 2z_G\ (z - z_G)$$

$$= h^2 + \rho^2 + 2x_G\ (x - x_G) + 2z_G\ (z - z_G)\ .$$

Therefore the moment of inertia is written

$$\int r^2 dm = h^2 \int dm + \int \rho^3 dm + 2x_G \int (x-x_G)\ dm + 2z_G \int (z-z_G)\ dm$$

Since $(x-x_G)$ and $(z-z_G)$ are the coordinates of dm with respect to the centre of gravity as the origin, we have from the definition of the centre of gravity,

$$\int (x - x_G)\ dm = 0,$$

$$\int (z - z_G)\ dm = 0,$$

$$\int r^2 dm = h^2 \int dm + \int \rho^2 dm \qquad (9)$$

where $\int \rho^2 dm$ is the moment of inertia of the body about the axis through G. Indicating the moment of inertia about the G-axis by k^2M, thus

$$\int \rho^2 dm = k^2 M$$

we have

$$\int r^2 dm = (h^2 + k^2)\ M \qquad (10)$$

k being called the radius of gyration about the axis through G.

Therefore (8) becomes

$$\frac{d^2\alpha}{dt^2}(h^2 + k^2)M = -gMh \sin\alpha$$

or

$$\frac{d^2\alpha}{dt^2}(h^2 + k^2) = -gh \sin\alpha \qquad (11)$$

This is the differential equation for the motion of the physical pendulum.

If we put

$$\frac{h^2 + k^2}{h} = \ell$$

$$\frac{d^2\alpha}{dt^2} + \frac{g}{\ell} \sin\alpha = 0 \qquad (12)$$

The differential equation is the same as (2) for a mathematical pendulum of length ℓ. We see that the physical pendulum has the same period as a mathematical pendulum of length.

$$\ell = \frac{h^2 + k^2}{h} \tag{13}$$

This is called the length of equivalent mathematical pendulum, or the reduced length of the physical pendulum.

If we take a parallel axis 0' at a distance 1 from the axis of suspension 0 at the farther side of the parallel axis G, the axis is called the axis of oscillation of the pendulum, here the three parallel axes being in the same plane.

The period T of the physical pendulum with amplitude d_o is given by

$$T = 2\pi \sqrt{\frac{\ell}{g}} \left(1 + \frac{\alpha^2}{16}\right)$$

when $d_o = 0$,

$$T_o = 2\pi \sqrt{\frac{\ell}{g}} = 2\pi \sqrt{\frac{\ell}{g} \cdot \frac{h^2 + k^2}{h}} \tag{14}$$

If the moment of inertia about 0 is indicated from (10) as

$$(h^2 + k^2) M = I ,$$

the required period is

$$T_o = 2\pi \sqrt{\frac{I}{Mgh}} \tag{15}$$

3. Corrections of Observed Period

For the determination of the period of a physical pendulum, the following five corrections are considered, thus 1) amplitude, 2) temperature, 3) effect of air 4) rate of chronometer, 5) flexure of the stand.

1) The correction for amplitude is to be corrected to the period of no amplitude by (5). When the amplitude is d_o, the period T_o of (14) corrected for d_o is given from (5)

$$T_o = T\,(1 - \frac{d_o^2}{16}) ,$$

where T is the observed period.

2) Concerning the effect of temperature, the temperature coefficient for the correction is determined for every pendulum.

3) As to the effect of air, there are three kinds to be taken into consideration: buoyancy, motion of air with the oscillating pendulum, and decrease of amplitude due to air resistance. On account of the buoyancy of the air, the weight of the pendulum diminishes from M to $gM - gM^1$, where M^1 is the air mass replaced by the pendulum.

If we assume that the centre of gravity G^1 of the replaced air is at a *distance of* h^1 *from* 0 *on the line connecting* 0 *and* G, (11) *becomes*

$$\frac{d^2\delta}{dt^2}\,(h^2 + k^2)\,M = -g\,(hM - h'M')\sin\alpha \qquad (16)$$

Here the left hand side shows the moment considered only from the motion of the pendulum regardless of the cause of the motion.

Putting

$$L = \frac{(h^2 + k^2)M}{hM - h'M'} = \frac{h^2 + k^2}{h(1 - \frac{h'M'}{hM})} \qquad (17)$$

we have

$$\frac{d^2\alpha}{dt^2} + g/\ell\ \mathrm{Sin}\,\alpha = C \qquad (18)$$

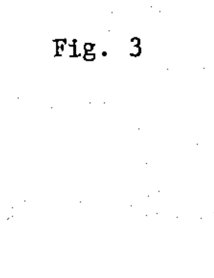


Fig. 3

This is the equation of motion of a mathematical pendulum with length ℓ oscillating in vacuum, as is shown in (1). That is to say, the period of the physical pendulum affected by the buoyancy due to air is the same as of the mathematical pendulum of length ℓ, defined by (17), which oscillates freely without the disturbance of air.

From (3) the period T of the pendulum with amplitude d_o is

$$T = 2\pi \sqrt{\frac{\ell}{g} \quad \frac{h^2 + k^2}{h} \quad \frac{1}{1 - \frac{h'M'}{hM}} \quad (1 + \frac{\alpha_o^2}{16})} \tag{19}$$

The period for $\alpha_o = 0$ is

$$T_o = T(1 - \frac{\alpha_o^2}{16}) \tag{20}$$

The effect of the buoyancy can be calculated by comparing (14) and (20). But the effect is usually taken into account with the effect of moving air mass considered below.

As to the effect of air mass moving with the pendulum, it can

$$\frac{d^2\alpha}{dt^2} (h^2 + k^2 + x^2 \frac{M'}{M}) M = -g (hM - h'M') \sin\alpha , \tag{21}$$

in which it is supposed that there is no gravitational effect due to the moving air mass.

Putting

$$\ell = \frac{h^2 + k^2 + x^2 \frac{M'}{M}}{h(1 - \frac{h'M'}{hM})} , \tag{22}$$

we have

$$\frac{d^2\alpha}{dt^2} + \frac{g}{\ell} \sin\alpha = 0$$

This is the equation of motion when both the effects of buoyancy and the moving air mass are considered. The period when $\alpha_o = 0$ is given by

$$To = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{h^2+k^2}{gh}\cdot\frac{1+\frac{x^2}{h^2+k^2}\ \frac{M'}{M}}{1-\frac{h'M'}{hM}}} \qquad (23)$$

If we put $x^2 = (h^2+k^2)K$, $\frac{M'}{M} = \delta$ and suppose $h' = h$,

$$To = 2\pi\left(\frac{h^2+k^2}{gh}\cdot\frac{1+k\delta}{1-\delta}\right)^{1/2} \qquad (24)$$

With the use of this equation, k can be determined empirically.

Two pendulums having the same form but different mass M_1 *and* M_2 *are used* and their period of oscillation T_1 and T_2 are determined at the same spot. Then

$$\frac{hg}{4\pi^2}\ T_1^2(1-\delta_1) = (h^2+k^2)(1+\delta_1 k)$$

$$\frac{hg}{4\pi^2}\ T_2^2(1-\delta_2) = (h^2+k^2)(1+\delta_2 k)\ ,$$

where $\delta_1 = \frac{M'}{M_1}$, $\delta_2 = \frac{M'}{M_2}$

$$\frac{T_1^2}{T_2^2} = \frac{1+\delta_1 K}{1+\delta_2 K},\ \frac{1-\delta_2}{1-\delta_1}\ ,$$

$$K = \frac{T_1^2(1-\delta_1) - T_2^2(1-\delta_2)}{\delta_1 T_2^2(1-\delta_2) - \delta_1 T_1^2\ (1-\delta_1)}\ ,$$

or

$$1+K = \frac{T_1^2 - T_2^2}{\frac{\delta_1 T_2^2}{1-\delta_1} - \frac{\delta_2 T_1^2}{1-\delta_2}} \qquad (25)$$

From this equation we can calculate K or x^2.

Let T air and T_o denote the period of the physical pendulum with the effect due to air (23) and the period with no such effect (14), the effect of the amplitude being corrected for both. Then the correction for the air which is to be substracted from the observed period T air is given by

$$T_{an} - T_o = 2\pi\left(\frac{h^2+k^2}{gh}\right)^{1/2}\left\{\left(\frac{1+k\delta}{1-\delta}\right)^{1/2} - 1\right\}$$

$$= 2\pi\left(\frac{h^2+k^2}{gh}\right)^{1/2}\left\{(1+\frac{1}{2}k\delta)(1+\frac{1}{2}\delta) - 1\right\}$$

$$= 2\pi \quad \frac{h^2+k^2}{gh} \quad \frac{\delta}{2}(1+k)$$

$$= T_o \frac{\delta}{2}(1+k)$$

$$= T_{an} \frac{\delta}{2}(1+k) \tag{26}$$

where $(1+k)$ is known empirically as is shown by (25).

As to h and $h^2 + k^2$, if the pendulum has a simple form, h can be measured and $h^2 + k^2$ can be calculated from its moment of inertia.

For $\delta = M'/M$ the density of the air mass M' can be determined by the air pressure in the pendulum container, the air pressure being measured with a manometer.

Concerning the effect of air resistance, it makes the amplitude of the oscillation of pendulum smaller. The resistance is supposed to be proportional to the angular velocity of the pendulum when α is small, and is given by how, which is a moment acting on the centre of gravity, c and w being respectively a proportional constant and the angular velocity of the pendulum.

Therefore if we consider the effect of this resistance, in addition to the effect due to g, the equation of motion (11) becomes

$$\frac{d^2\alpha}{dt^2}(h^2+k^2) = -gh \sin\alpha$$

$$\therefore \frac{d^2\alpha}{dt^2} + \frac{c}{\ell}\frac{d\alpha}{dt} + g/\ell\,\alpha = 0 \tag{27}$$

where we put $\ell = \dfrac{h^2+k^2}{h}$ and $\sin\alpha = \alpha$

This is the differential equation of the physical pendulum when we take the air resistance into consideration. If we solve this t homogeneous differential equation of the 2nd order, we have the amplitude as a function of time t. And we can find the decrease of amplitude with the lapse of time.

In the case of practical measurement the inside of the pendulum container is kept in high vacuum. Therefore the effect of resistance is much reduced and practically no special correction is given except that the decrease of the amplitude is measured and is considered for the amplitude correction.

4) Rate of chronometer change

The change of rate is determined compared with the radio time signal up to the accuracy of 0.01 sec./day. If one second of the chronometer used is equal to accurate $(1+c)$ second, the period T^1 sec. determined with the chronometer should be corrected to $T = T'\ (1+c)$ sec. (28)
When the chronometer gains, t is negative and vice versa.

5) Concerning the motion of the pendulum support caused by the oscillation of the pendulum, we suppose that the axis of oscillation displaces f in time t toward the positive direction of x-axis with the angular displacement α of the pendulum from the vertical position (Fig. 2).

Instead of (8) we have in this case $\frac{dw}{dt}\int r^2 dm = -gMh\ \sin\alpha + \frac{d^2f}{dt^2} Mh$ $\cos\alpha$,
where the left hand side is the moment considered from the motion of the pendulum and the right hand side is the moment due to the external forces causing the motion of the pendulum. So from (10)

$$\frac{d^2\alpha}{dt^2}(h^2+k^2) = -gh\ \sin\alpha + \frac{d^2f}{dt^2} h\ \cos\alpha$$

If we put $L = \frac{h^2+k^2}{h}$

we have $$\frac{d^2\alpha}{dt^2} + \frac{g}{\ell}\sin\alpha - \frac{1}{\ell}\frac{d^2f}{dt^2}\cos\alpha = 0 \qquad (29)$$

This is the differential equation of the pendulum motion when the flexure of the support is taken into account, and ℓ is the length of equivalent mathematical pendulum when there is no horizontal acceleration as $\frac{d^2f}{dt^2}$

The horizontal acceleration $\frac{d^2f}{dt^2}$ in (29) can be expressed as a function of as follows. Let the horizontal displacement of dm be dx in time dt. Then its horizontal velocity dx/dt is given by

$$\frac{dx}{dt} = V\frac{2}{r} + \frac{df}{dt} = Z\,W + \frac{df}{dt}$$

where v is the velocity of dm directed to the motion. Therefore

$$\frac{d^2x}{dt^2} = Z\frac{dw}{dt} + w\frac{dz}{dt} + \frac{d^2f}{dt^2}$$

$$= Z\frac{dw}{dt} + x\,w^2 + \frac{d^2f}{dt^2}, \qquad \frac{dx}{dt} = V\frac{2}{r}$$

The horizontal force F due to the whole mass is

$$F = \int\frac{d^2x}{dt^2}\,dm = \frac{dw}{dt}Z_GM + W^2xGM + \frac{d^2f}{dt^2}M \qquad (30)$$

The elastic force of the support due to the above force F is expressed by ϵf, where ϵ is the elastic constant of the support or 1/E is numerically equal to the displacement f of the support when the external force F is unity ϵf acts in the opposite direction of F.

$$\epsilon f + F = 0 \qquad (31)$$

F of (30) can be simplied. From (11)

$$\frac{d^2\alpha}{dt^2} = -g/\ell\ \sin\alpha \qquad (32)$$

But from the description in 1.

$$\left(\frac{d\delta}{dt}\right)^2 = 2\,g/\ell\,(\cos\alpha - \cos\alpha_o)$$
$$= g/\ell\,(\sin^2\alpha_o - \sin^2\alpha) \qquad (33)$$

$$\left[\therefore \cos A = (1 - \sin^2 A)^{1/2} = 1 - \frac{1}{2}\sin^2 A + \ldots\ldots\right]$$

Substituting (32) and (33) into the right hand side of (30), we have

$$F = \frac{hg}{\ell} M\,(\sin\alpha\cos\alpha + \sin\alpha\sin^2\alpha_o - \sin^3\alpha),$$

where $h\cos\alpha = 2_\alpha$, $h\sin\alpha = x_G$, and the last term of (30) is neglected. If we also neglect the terms like $\sin^3\alpha$ and take α for $\sin\alpha$,

$$F = \frac{hg}{\ell} M\alpha \qquad (34)$$

Therefore (31) becomes

$$f = -\frac{hg}{\varepsilon\ell} M\alpha \qquad (35)$$

or

$$f = -\gamma_\alpha$$

where

$$\gamma = \frac{hg}{\varepsilon\ell} M \qquad (36)$$

Thus f is expressed as a function of d. Putting the value of f into (29),

$$\frac{d^2\alpha}{dt^2} + \frac{g}{\ell}\sin\alpha + \frac{hg}{\varepsilon\ell^2} M\cos\alpha\frac{d^2\alpha}{dt^2} = 0$$

or

$$\frac{d^2\alpha}{dt^2}\left(1 + \frac{hg}{\varepsilon\ell^2} M\right) + \frac{g}{\ell}\sin\alpha = 0 \qquad (37)$$

where ℓ is taken for $\cos\alpha$.

Comparing (37) with (12), we find that the equivalent length of the mathematical pendulum ℓ' affected by the motion of the support is given by

$$\ell' = \ell\left(1 + \frac{hg}{\varepsilon\ell^2} M\right) \qquad (35)$$

Therefore the equivalent length of the mathematical pendulum increases by

$$\Delta \ell = \frac{Mgh}{\varepsilon \ell}$$

owing to the oscillation of the support.

The increase of period ΔT is given as follows:

$$T + \Delta T = 2\pi \sqrt{\ell'/g} = 2\pi \sqrt{\ell/g} \sqrt{1 + \frac{hg}{\varepsilon \ell^2} M}$$

$$= 2\pi \sqrt{\ell/g} \; (1 + \frac{1}{2} \frac{hg}{\varepsilon \ell^2} M)$$

$$\Delta T = T \frac{Mgh}{2\varepsilon \ell^2} \qquad (39)$$

Here the observed period can be used for T.

The correction ΔT is always to be substracted from the observed period, ε can be found with the use of (31) by applying a definite horizontal force to the pendulum support with a weight through a pulley and measuring the displacement of the support.

4. Reversible Pendulum:

In a physical pendulum the axis of suspension 0 and the axis of oscillation 0' written in 2, are interchangeable. That is to say, if we oscillate the physical pendulum about the axis 0' which is parallel to the axis 0, 0' becomes the axis of suspension and 0 the axis of oscillation, and the period of the oscillation is the same as that when 0 is the axis of suspension (Fig. 9).

Therefore when the physical pendulum oscillates about 0' its length ℓ' of mathematical pendulum is equal to 1. This is shown as follows. From 1 of (12).

$$\ell' = \frac{(\ell - h)^2 + k^2}{\ell - h} = (\ell - h) + \frac{k^2}{\ell - h}$$

While

$$\ell = h + \frac{k^2}{h} , \quad \text{or} \qquad \ell - h = \frac{k^2}{h}$$

$$\frac{k^2}{\ell - h} = h$$

Therefore

$$\ell' = (\ell - h) + h = \ell$$

The pendulum of which the two axes of oscillation can be interchangeably used is called the reversible pendulum. To determine the exact positions of the two axes in a physical pendulum is very difficult. So we fix them approximately at first and then give corrections to fulfil the requirements.

We take two parallel axes in a plane passing the centre of gravity, and determine the length of equivalent mathematical pendulum for each axis. Let T, I and h be the period, the reduced length and the distance of G from the axis of suspension respectively, and their suffix, 1 and 2, be respectively for the first axis and the second axis. Then

$$T_1 = 2\pi \sqrt{\frac{\ell_1}{g}} , \quad T_2 = 2\pi \sqrt{\frac{\ell_2}{g}}$$

When the two axes are exactly interchangeable, the period which we search for is given by

$$T = 2\pi \sqrt{\frac{\ell}{g}}$$

$$\frac{T^2}{T_1^2} = \frac{\ell}{\ell_1} = \frac{h_1 + h_2}{h_1 + \frac{k^2}{h_1}}$$

$$\frac{T^2}{T_2^2} = \frac{\ell}{\ell_2} = \frac{h_1 + h_2}{h_2 + \frac{k^2}{h_2}} \tag{40}$$

where

$$\ell = h_1 + h_2$$

$$\ell_1 = h_1 + \frac{k^2}{h_1}$$

$$\ell_2 = h_2 + \frac{k^2}{h_2}$$

$$\therefore \quad T_1^2 (h_1 + h_2) = h_1 T^2 + \frac{k^2}{h_1} T^2$$

$$T_2^2 (h_1 + h_2) = h_2 T^2 + \frac{k^2}{h_2} T^2$$

$$\therefore \quad (h_1 + h_2)(h_1 T_1^2 - h_2 T_2^2) = (h_1^2 - h_2^2) T^2$$

$$\therefore \quad T^2 = \frac{h_1 T_1^2 - h_2 T_2^2}{h_1 - h_2}$$

The right hand side of this equation is simplified as follows:

Putting $\quad T_1 = T_2 + \delta T ,$

$$T^2 = \frac{h_1 T_1^2 - h_2 (T_1^2 - 2T_1 \delta T)}{h_1 - h_2}$$

$$= T_1^2 \left(1 + \frac{2h_2}{h_1 - h_2} \quad \frac{\delta T}{T_1}\right)$$

$$T = T_1 + \frac{h_2 \quad \delta T}{h_1 - h_2}$$

Likewise $\quad T^2 = \dfrac{h_1 (T_2^2 + 2T_2 \delta T) - h_2 T_2^2}{h_1 - h_2}$

$$= T_2 \quad \left(1 + \frac{2h_1}{h_1 - h_2} \quad \frac{\delta T}{T_2}\right)$$

$$T = T + \frac{h_1}{h_1 - h_2} \quad dT$$

From the above two forms of T,

$$T = \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{2} \; \frac{h_1 + h_2}{h_1 - h_2} \tag{42}$$

The second term is a correction to the mean of the two observed periods. This increases with the difference of the two periods and decreases with the difference of the two distances of O and O' from the center of gravity. So the reversible pendulum is made so that $T_1 - T_2$ becomes smaller and $h_1 - h_2$ is as large as possible.

For this purpose, two knife edges are fixed first, i.e. $h_1 + h_2$ is made constant, and then the position of G is changed by a movable weight w so that T_1 and T_2 take close values. W is used for changing $h_1 - h_2$ largely.

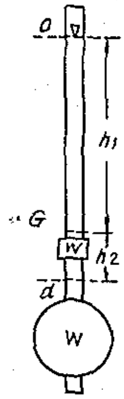


Fig. 4

As to the effect of air, it becomes much simpler if the form of the pendulum is symmetrical with respect to its geometrical centre. The centre of gravity G' of the air replaced by the pendulum is at the middle point between the two axes O and O'. i.e., the distance h_1 between O and G' is equal to the distance h_2 between O' and G'. Denoting the distance by h', we have from (22) which gives the reduced length of the physical pendulum when the effect of air is taken into consideration,

$$\ell_1 = \frac{h_1^2 + k^2 + x^2 \frac{M'}{M}}{h_1 \left(1 - \frac{h'M'}{h_1 M}\right)} = \frac{h_1 M + k^2 M + x^2 M'}{h_1 M - h'M'} ,$$

$$\ell_2 = \frac{h_2 + k^2 + x^2 \frac{M'}{M}}{h_2 \left(1 - \frac{h'M'}{h_2 M}\right)} = \frac{h_2 M + k^2 M + x^2 M'}{h_2 M - h'M'}$$

Subtracting the above two equations each other,

$$(\ell_1 h_1 - \ell_2 h_2) M - (\ell_1 - \ell_2) h'M' = (h_1^2 - h_2^2) M$$

While, from (40)

$$\ell_1 = (h_1 + h_2) \frac{T_1^2}{T^2} ,$$

$$\ell_2 = (h_1 + h_2) \frac{T_2^2}{T^2}$$

Substituting these ℓ_1 and ℓ_2 into the above equation,

$$\frac{h_1 + h_2}{T^2} (h_1 T_1^2 - h_2 T_2^2) M - \frac{h_1 + h_2}{T^2} (T_1^2 - T_2^2) h'M' = (h_1^2 - h_2^2) M$$

$$T^2 = \frac{h_1 T_1^2 - h_2 T_2^2}{h_1 - h_2} - \frac{(T_1^2 - T_2^2) h'M'}{h_1 - h_2}$$

For this purpose, two knife edges are fixed first, i.e. $h_1 + h_2$ is made constant,

By the calculation similar to (42), we have

$$T = \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{2} + \frac{h_1 + h_2}{h_1 - h_2} - \frac{T_1 - T_2}{2} \frac{h'M'}{h_1 - h_2} \qquad (43)$$

The last term is the effect of air, which is also proportional to $(T_1 - T_2)/(h_1 - h_2)$, and in this case it is not necessary to know k^2. In order to minimize the effect of air, the reversible pendulum is usually made so that the outer form of the pendulum has symmetry on either side of the geometrical centre and the centre of gravity is deflected to one side of the geometrical centre as far as possible.

The reversible pendulum has been used for the absolute measurement of gravity.

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